WEIGHTED ARADHANA DISTRIBUTION: PROPERTIES AND APPLICATIONS

Rashid A. Ganaie, V. Rajagopalan and Aafaq A. Rather

Department of Statistics, Annamalai University, Annamalai nagar, Tamil Nadu

1 Email: rashidan7745@gmail.com, 2 Email: auvr64@yahoo.com 3 Corresponding Author Email: aafaq7741@gmail.com

Abstract

In this paper, we introduced a new generalization of Aradhana distribution called as weighted Aradhana distribution (WAD), for modeling life-time data. This has been compared with one parameter Aradhana distribution. The statistical properties of this distribution has been derived and the model parameters are estimated by maximum likelihood estimation. Finally, an application to real life-time data set is fitted and the fit has been found to be good.

Keywords: Aradhana distribution, weighted distribution, Maximum Likelihood Estimator, Likelihood Ratio test, order statistics.

1. Introduction

The idea of weighted distributions was first given by Fisher (1934) to model the ascertainment bias. Later Rao (1965) has developed this concept in a unified manner while modeling the statistical data, when the standard distributions were not appropriate to record these observations with equal probabilities. As a result, weighted models were formulated in such situations to record the observations according to some weighted function. The weighted distribution reduces to length biased distribution as the weight function considers only the length of the units. The concept of length biased sampling, was first introduced by Cox (1969) and Zelen (1974). Van Deusen in (1986) arrived at Size-biased distribution theory independently and fit it to the distributions of diameter of breast height (DBH). Subsequently, Lappi and Bailey (1987), used weighted distributions to analyse the HPS diameter increment data. In fisheries, Taillie et al (1995) modeled populations of fish stocks using weights. Generally, the size-biased distribution is when the sampling mechanism selects the units with probability which is proportional to some measure of the unit size. There are various good sources which provide the detailed description of weighted distributions. Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different fields. Weighted distributions are applied in various research areas related to biomedicine, reliability, ecology and branching processes. Para and Jan (2018) introduced the Weighted Pareto type-II distribution as a new model for handling medical science data and studied its statistical properties and applications. Rather and Subramanian (2018) discussed the characterization and estimation of length biased weighted generalized uniform distribution. Rather and Subramanian (2019), discussed the weightedushila distribution with its various statistical properties and applications.
Aradhana distribution is a newly proposed lifetime model formulated by Shanker (2016) for several medical applications and calculated its various mathematical and statistical properties including its shape, moment, stochastic ordering, generating function, skewness, kurtosis, mean deviation, Renyi entropy, order statistics, stress-strength reliability, hazard rate function, mean residual life function Bonferroni and lorenz curves, Maximum likelihood estimation. The new one parameteric life-time distribution called as Aradhana distribution has better flexibility in handling lifetime data as compared to Lindley and exponential distributions. Shankar (2016) introduced two parameter quasi Aradhana distribution and obtained its raw moments and central moments and also discussed its mathematical and statistical properties. Shankar (2017) has obtained a poisson Aradhana distribution and named as discrete poisson-Aradhana distribution discussed its various statistical properties and estimation of parameter.

2. Weighted Aradhana Distribution

The probability density function of the Aradhana distribution is given by

\[
f(x; \theta) = \frac{\theta^3}{(\theta^2 + 2\theta + 2)}(1 + x)^2 e^{-\theta x}; \quad x > 0, \theta > 0
\]  

and the cumulative density function of the Aradhana distribution is given by

\[
F(x; \theta) = 1 - \left[1 + \frac{\theta x(\theta x + 2\theta + 2)}{(\theta^2 + 2\theta + 2)}\right] e^{-\theta x}; \quad x > 0, \theta > 0
\]  

Suppose \(X\) is a non-negative random variable with probability density function \(f(x)\). Let \(w(x)\) be the non-negative weight function, then the probability density function of the weighted random variable \(X_w\) is given by

\[
f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \quad x > 0,
\]

Where \(w(x)\) be a non-negative weight function and \(E(w(x)) = \int w(x)f(x)dx < \infty\).

In this paper, we will consider the weight function as \(w(x) = x^c\) to obtain the weighted Aradhana distribution. The probability density function of weighted Aradhana distribution is given as:

\[
f_{w^c}(x) = \frac{x^c f(x)}{E(x^c)}; \quad x > 0
\]
Substitute (1) and (4) in equation (3), we will get the required probability density function of weighted Aradhana distribution as

\[ f_w(x) = \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} x^c (1+x)^2 e^{-\theta x} \]  

(5)

Now, the cumulative density function (cdf) of the weighted Aradhana distribution (WAD) is obtained as

\[ F_w(x) = \int_0^x f_w(x) dx \]

\[ = \int_0^x \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} x^c (1+x)^2 e^{-\theta x} dx \]

After simplification, we will get the cumulative distribution function of weighted Aradhana distribution (WAD) as

\[ F_w(x) = \frac{\theta^2}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \left( \gamma(c+1, \theta x) + \gamma(c+3, \theta x) + 2\theta \gamma(c+2, \theta x) \right) \]  

(6)

3. Reliability Analysis

In this section, we will discuss about the survival function, failure rate, reverse hazard rate and the Mills ratio of the weighted Aradhana distribution (WAD).
The survival function or the reliability function of the weighted Aradhana distribution is given by

\[ S_w(x) = 1 - \frac{\theta^2}{(\theta^2 \Gamma(c + 1) + \Gamma(c + 3) + 2\theta \Gamma(c + 2))} \left( \gamma(c + 1, \theta x) + \gamma(c + 3, \theta x) + 2\theta \gamma(c + 2, \theta x) \right) \]

The hazard function is also known as the hazard rate, instantaneous failure rate or force of mortality and is given by

\[ h(x) = \frac{f_w(x)}{1 - F_w(x)} \]

\[ = \frac{\theta^{c+3}}{(\theta^2 \Gamma(c + 1) + \Gamma(c + 3) + 2\theta \Gamma(c + 2)) - (\theta^2 \gamma(c + 1, \theta x) + \gamma(c + 3, \theta x) + 2\theta \gamma(c + 2, \theta x))} x^c (1 + x)^2 e^{-\theta x} \]

The Reverse hazard rate is given by

\[ h_r(x) = \frac{\theta^{c+3}}{(\theta^2 \gamma(c + 1, \theta x) + \gamma(c + 3, \theta x) + 2\theta \gamma(c + 2, \theta x))} x^c (1 + x)^2 e^{-\theta x} \]

And the Mills Ratio of the weighted Aradhana distribution is

\[ \text{Mills Ratio} = \frac{1}{h_r(x)} = \frac{(\theta^2 \gamma(c + 1, \theta x) + \gamma(c + 3, \theta x) + 2\theta \gamma(c + 2, \theta x))}{\theta^{c+3} x^c (1 + x)^2 e^{-\theta x}} \]

![Graph](image-url)

4. Moments and Associated Measures

Let \( X \) denotes the random variable of weighted Aradhana distribution with parameters \( \theta \) and \( c \), then the \( r^{th} \) order moment \( E(X^r) \) of weighted Aradhana distribution is obtained as
\[ E(X^r) = \mu_r = \int_0^\infty x^r f_{xw}(x)dx \]

\[ = \int_0^\infty x^r \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta\Gamma(c+2))} (1+x)^2 e^{-\theta x} dx \]

\[ = \int_0^\infty \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta\Gamma(c+2))} (1+x)^2 e^{-\theta x} dx \]

\[ \Rightarrow E(X^r) = \frac{(\theta^2 \Gamma(c+r+1) + \Gamma(c+r+3) + 2\theta\Gamma(c+r+2))}{\theta^r (\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta\Gamma(c+2))} \tag{7} \]

Putting \( r = 1 \) in equation (7), we will get the mean of weighted Aradhana distribution which is given by

\[ E(X) = \mu_1 = \frac{(\theta^2 \Gamma(c+2) + \Gamma(c+4) + 2\theta\Gamma(c+3))}{\theta (\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta\Gamma(c+2))} \]

and putting \( r = 2 \) in equation (7), we get the second moment of weighted Aradhana distribution as

\[ E(X^2) = \frac{(\theta^2 \Gamma(c+3) + \Gamma(c+5) + 2\theta\Gamma(c+4))}{\theta^2 (\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta\Gamma(c+2))} \]

Therefore,

\[ \text{Variance} = \left( \frac{(\theta^2 \Gamma(c+3) + \Gamma(c+5) + 2\theta\Gamma(c+4))}{\theta^2 (\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta\Gamma(c+2))} \right) - \left( \frac{(\theta^2 \Gamma(c+2) + \Gamma(c+4) + 2\theta\Gamma(c+3))}{\theta (\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta\Gamma(c+2))} \right)^2 \]

4.1 Harmonic mean

The Harmonic mean of the proposed model can be obtained as

\[ H.M = E\left( \frac{1}{x} \right) = \int_0^\infty \frac{1}{x} f_{xw}(x)dx \]
\[
\begin{align*}
&= \int_0^{\infty} \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} x^c (1+x)^2 e^{-\theta x} \, dx \\
&= \int_0^{\infty} \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} x^{c+1} (1+x)^2 e^{-\theta x} \, dx \\
&= \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \left( \int_0^{\infty} x^{c+1} (1+x^2 + 2x) e^{-\theta x} \, dx \right) \\
&= \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \left( \int_0^{\infty} x^{(c+1)-2} e^{-\theta x} \, dx + \int_0^{\infty} x^{(c+2)-1} e^{-\theta x} \, dx + 2 \int_0^{\infty} x^{(c+1)-1} e^{-\theta x} \, dx \right) \\
\Rightarrow H.M &= \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \left( \gamma(c+1, \theta x) + \gamma(c+2, \theta x) + 2\gamma(c+1, \theta x) \right) \\
\end{align*}
\]

4.2 Moment generating function and characteristics function of weighted Aradhana distribution

Let \( X \) have a weighted Aradhana distribution, then the MGF of \( X \) is obtained as

\[
M_X(t) = E(e^{itx}) = \sum_{j=0}^{\infty} e^{itx} f_w(x) \, dx
\]

Using Taylor’s series

\[
M_X(t) = E(e^{itx}) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^{\infty} x^j f_w(x) \, dx
\]

\[
= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j,
\]

\[
= \sum_{j=0}^{\infty} \frac{t^j}{j!} \left( \frac{\theta^2 \Gamma(c+j+1) + \Gamma(c+j+3) + 2\theta \Gamma(c+j+2)}{\theta \Gamma^2(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)} \right)
\]

\[
\Rightarrow M_X(t) = \frac{1}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \sum_{j=0}^{\infty} \frac{t^j}{j!} \theta^j \left( \theta^2 \Gamma(c+j+1) + \Gamma(c+j+3) + 2\theta \Gamma(c+j+2) \right)
\]

Similarly, the characteristics function of weighted Aradhana distribution can be obtained as

\[
\varphi_X(t) = M_X(it)
\]
\[ M_s (it) = \frac{1}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \sum_{j=0}^{n} \frac{(it)^j}{j! \theta^j} (\theta^2 \Gamma(c+j+1) + \Gamma(c+j+3) + 2\theta \Gamma(c+j+2)) \]

5. Order Statistics

Let \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) be the order statistics of a random sample \( X_1, X_2, \ldots, X_n \) drawn from the continuous population with probability density function \( f_x(x) \) and cumulative density function with \( F_x(x) \), then the probability density function of \( r^{th} \) order statistics \( X_{(r)} \) is given by

\[
f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F_x(x)^{r-1} (1 - F_x(x))^{n-r} \]

Using the equations (5) and (6) in equation (8), the probability density function of \( r^{th} \) order statistics \( X_{(r)} \) of weighted Aradhana distribution is given by

\[
f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \left( \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} x^c (1+x)^2 e^{-\theta x} \right) \]

\[
\times \left( \frac{\theta^2}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \left( \gamma(c+1, \theta x) + \gamma(c+3, \theta x) + 2\theta \gamma(c+2, \theta x) \right) \right)^{r-1} \]

\[
\times \left( 1 - \frac{\theta^2}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \left( \gamma(c+1, \theta x) + \gamma(c+3, \theta x) + 2\theta \gamma(c+2, \theta x) \right) \right)^{n-r} \]

Therefore, the probability density function of higher order statistics \( X_{(n)} \) can be obtained as

\[
f_{X_{(n)}}(x) = \left( \frac{n \theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} x^c (1+x)^2 e^{-\theta x} \right) \]

\[
\times \left( \frac{\theta^2}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \left( \gamma(c+1, \theta x) + \gamma(c+3, \theta x) + 2\theta \gamma(c+2, \theta x) \right) \right)^{n-1} \]

And therefore the probability density function of \( l^{th} \) order statistics \( X_{(l)} \) can be obtained as
6. Likelihood Ratio Test

Let \( X_1, X_2, \ldots, X_n \) be a random sample from the weighted Aradhana distribution. To test the hypothesis

\[
H_0 : f(x) = f(x; \theta) \quad \text{against} \quad H_1 : f(x) = f_w(x;c, \theta)
\]

In order to test whether the random sample of size \( n \) comes from the Aradhana distribution or weighted Aradhana distribution, the following test statistic is used

\[
\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \left( \frac{f_w(x_i;c, \theta)}{f(x; \theta)} \right)
\]

\[
= \prod_{i=1}^{n} \left( \frac{\theta^c (\theta^2 + 2\theta + 2)x_i^c}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \right)
\]

\[
= \left( \frac{\theta^c (\theta^2 + 2\theta + 2)}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \right)^n \prod_{i=1}^{n} x_i^c
\]

Thus, we reject the null hypothesis if

\[
\Delta = \left( \frac{\theta^c (\theta^2 + 2\theta + 2)}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \right)^n \prod_{i=1}^{n} x_i^c > k
\]

or,

\[
\Delta^* = \prod_{i=1}^{n} x_i^c > \left( \frac{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))}{\theta^c (\theta^2 + 2\theta + 2)} \right)^n
\]

\[
\Delta^* = \prod_{i=1}^{n} x_i^c > k^*, \text{ where } k^* = k \left( \frac{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))}{\theta^c (\theta^2 + 2\theta + 2)} \right)^n
\]

Thus, for large sample size \( n \), \( 2 \log \Delta \) is distributed as chi-square distribution with one degree of freedom and also p-value is obtained from the chi-square distribution. Thus we reject the null hypothesis, when the probability value is given by
\[ p(\Delta^* > \beta^*) \text{, where } \beta^* = \prod_{i=1}^{n} x_i^* \text{is less than the specified level of significance and } \prod_{i=1}^{n} x_i^c \]

Is the observed value of the statistic \( \Delta^* \).

7. Maximum Likelihood Estimator and Fisher Information Matrix

This is one of the most useful methods for estimating the different parameters of the distribution. Let \( X_1, X_2, \ldots, X_n \) be the random sample of size \( n \) drawn from the weighted Aradhana distribution, then the likelihood function of weighted Aradhana distribution is given as:

\[
L(x, c, \theta) = \prod_{i=1}^{n} f_w(x; c, \theta) \\
= \prod_{i=1}^{n} \left( \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} x_i^c (1 + x_i)^2 e^{-\theta x_i} \right) \\
L(x, c, \theta) = \frac{\theta^{n(c+3)}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))^n} \prod_{i=1}^{n} x_i^c (1 + x_i)^2 e^{-\theta x_i}
\]

The log likelihood function is

\[
\log L = n(c+3) \log \theta - n \log \left( \theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2) \right) + c \sum_{i=1}^{n} \log x_i \\
+ 2 \sum_{i=1}^{n} \log(1 + x_i) - \theta \sum_{i=1}^{n} x_i
\]

(9)

The maximum likelihood estimates of \( \theta \) and \( c \) can be obtained by differentiating equation (9) with respect to \( \theta \) and \( c \) and must satisfy the normal equation

\[
\frac{\partial \log L}{\partial \theta} = \frac{n(c+3)}{\theta} - n \left( \frac{2\Gamma(c+1) + 2\Gamma(c+2)}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \right) - \sum_{i=1}^{n} x_i = 0
\]

\[
\frac{\partial \log L}{\partial c} = n \log \theta - n \psi(c+1) + \sum_{i=1}^{n} \log x_i = 0
\]

Where \( \psi(.) \) is the digamma function.
Because of the complicated form of the likelihood equations, algebraically it is very difficult to solve the system of non-linear equations. Therefore we use R and wolfram mathematics for estimating the required parameters. To obtain confidence interval we use the asymptotic normality tests. We have that, if \( \hat{\lambda} = (\hat{\theta}, \hat{c}) \) denotes the Maximum likelihood estimates of \( \lambda = (\theta, c) \), we can state the result as follows:

\[
\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_2(0, I^{-1}(\lambda))
\]

Where \( I(\lambda) \) is Fisher’s Information Matrix, i.e.,

\[
I(\lambda) = -\frac{1}{n} \begin{pmatrix}
E \left( \frac{\partial^2 \log L}{\partial \theta^2} \right) & E \left( \frac{\partial^2 \log L}{\partial \theta \partial c} \right) \\
E \left( \frac{\partial^2 \log L}{\partial c \partial \theta} \right) & E \left( \frac{\partial^2 \log L}{\partial c^2} \right)
\end{pmatrix}
\]

Where

\[
E \left( \frac{\partial^2 \log L}{\partial \theta^2} \right) = -\frac{n(c + 3)}{\theta^2} - n \left( \frac{\theta^2 \Gamma(c + 1) + \Gamma(c + 3) + 2\theta \Gamma(c + 2) + 2\Gamma(c + 1) - (2\theta \Gamma(c + 1) + 2\Gamma(c + 2))}{(\theta^2 \Gamma(c + 1) + \Gamma(c + 3) + 2\theta \Gamma(c + 2))^2} \right)
\]

\[
E \left( \frac{\partial^2 \log L}{\partial c} \right) = -n \psi'(c + 1)
\]

\[
E \left( \frac{\partial^2 \log L}{\partial \theta \partial c} \right) = \frac{n}{\theta}
\]

where \( \psi(.)' \) is the first order derivative of digamma function.

since \( \lambda \) being unknown, we estimate \( I^{-1}(\lambda) \) by \( I^{-1}(\hat{\lambda}) \) and this can be used to obtain asymptotic confidence intervals for \( \theta \) and \( c \)

8. Bonferroni And Lorenz Curves

The Bonferroni and the Lorenz curves are not only used in economics in order to study the income and poverty, but it is also being used in other fields like reliability, medicine, insurance and demography. The Bonferroni and Lorenz curves are given by

\[
B(p) = \frac{1}{p \mu_1} \int_0^p xf(x)dx
\]
and \( L(p) = \frac{1}{\mu_1} \int_0^q x f(x) dx \)

where \( \mu_1 = \frac{(\theta^2 \Gamma(c + 2) + \Gamma(c + 4) + 2\theta \Gamma(c + 3))}{\theta(\theta^2 \Gamma(c + 1) + \Gamma(c + 3) + 2\theta \Gamma(c + 2))} \) and \( q = F^{-1}(p) \)

\[
B(p) = \frac{\theta^{c+4}}{p(\theta^2 \Gamma(c + 2) + \Gamma(c + 4) + 2\theta \Gamma(c + 3))} \left( \gamma(c + 2, \theta q) + \gamma(c + 4, \theta q) + 2\gamma(c + 3, \theta q) \right)
\]

After simplification, we get

\[
B(p) = \frac{\theta^{c+4}}{(\theta^2 \Gamma(c + 2) + \Gamma(c + 4) + 2\theta \Gamma(c + 3))} \left( \gamma(c + 2, \theta q) + \gamma(c + 4, \theta q) + 2\gamma(c + 3, \theta q) \right)
\]

And

\[
L(p) = pB(p) = \frac{\theta^{c+4}}{(\theta^2 \Gamma(c + 2) + \Gamma(c + 4) + 2\theta \Gamma(c + 3))} \left( \gamma(c + 2, \theta q) + \gamma(c + 4, \theta q) + 2\gamma(c + 3, \theta q) \right)
\]

### 9. Entropies

The concept of entropy is important in different areas such as probability and statistics, physics, communication theory and economics. Entropies quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable \( X \) is a measure of variation of the uncertainty.

#### 9.1: Renyi Entropy

The Renyi entropy is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

\[
e(\beta) = \frac{1}{1 - \beta} \log \left( \int f^\beta(x) dx \right)
\]

where, \( \beta > 0 \) and \( \beta \neq 1 \)

\[
e(\beta) = \frac{1}{1 - \beta} \log \left( \int_0^{\infty} \left( \frac{\theta^{c+3}}{(\theta^2 \Gamma(c + 1) + \Gamma(c + 3) + 2\theta \Gamma(c + 2))} \right) x^{c+1}(1 + x)^{2\beta} e^{-\theta x} dx \right)
\]

\[
e(\beta) = \frac{1}{1 - \beta} \log \left( \int_0^{\infty} \left( \frac{\theta^{c+3}}{(\theta^2 \Gamma(c + 1) + \Gamma(c + 3) + 2\theta \Gamma(c + 2))} \right) x^{c+1}(1 + x)^{2\beta} e^{-\theta x} dx \right)
\]
$$e(\beta) = \frac{1}{1-\beta} \log \left( \frac{\theta^{c+3}}{\left(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)\right)} \sum_{j=0}^{\infty} \left( \beta \right)^j \int_0^\infty x^j e^{-\beta_0 x} (1+x)^{2\beta} \, dx \right)$$  (10)

Using binomial expansion in equation (10), we get

$$e(\beta) = \frac{1}{1-\beta} \log \left( \frac{\theta^{c+3}}{\left(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)\right)} \sum_{j=0}^{\infty} \left( \beta \right)^j \int_0^\infty e^{-\beta_0 x} x^{j(c+1)-1} \, dx \right)$$

$$e(\beta) = \frac{1}{1-\beta} \log \left( \frac{\theta^{c+3}}{\left(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)\right)} \sum_{j=0}^{\infty} \left( \beta \right)^j \frac{\Gamma(\beta c + j + 1)}{(\beta \theta)^{j+1}} \right)$$

9.2: Tsallis Entropy

A generalization of Boltzmann-Gibbs (B-G) statistical mechanics initiated by Tsallis has focused a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows

$$S_\lambda = \frac{1}{\lambda - 1} \left( 1 - \int_0^\infty f^\lambda (x) \, dx \right)$$

$$S_\lambda = \frac{1}{\lambda - 1} \left( 1 - \int_0^\infty \left( \frac{\theta^{c+3}}{\left(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)\right)} x^c (1+x)^2 e^{-\theta x} \right)^\lambda \, dx \right)$$

$$S_\lambda = \frac{1}{\lambda - 1} \left( 1 - \int_0^\infty \left( \frac{\theta^{c+3}}{\left(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)\right)} \right)^\lambda \int_0^\infty x^{2c} e^{-2\theta_0 x} (1+x)^{2\lambda} \, dx \right)$$  (11)

Using binomial expansion in equation (11), we get

$$S_\lambda = \frac{1}{\lambda - 1} \left( 1 - \left( \frac{\theta^{c+3}}{\left(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)\right)} \right)^\lambda \sum_{j=0}^{\infty} \left( \lambda \right)^j \int_0^\infty x^j e^{-\lambda \theta_0 x} \, dx \right)$$

$$S_\lambda = \frac{1}{\lambda - 1} \left( 1 - \left( \frac{\theta^{c+3}}{\left(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2)\right)} \right)^\lambda \sum_{j=0}^{\infty} \left( \lambda \right)^\infty \int_0^{\lambda (c+1)-1} e^{-\lambda \theta_0 x} \, dx \right)$$
\[ S_{j} = \frac{1}{\lambda - 1} \left( 1 - \frac{\theta^{c+3}}{(\theta^2 \Gamma(c+1) + \Gamma(c+3) + 2\theta \Gamma(c+2))} \sum_{j=0}^{\infty} \left( \frac{\lambda}{\lambda \theta} \right)^{c+j+1} \right) \]

10. Data Analysis

The weighted Aradhana distribution has been fitted to a number of life-time data set from biomedical science and Engineering. In this section, we present a real life-time data set to show that weighted Aradhana distribution can be better than one parametric Aradhana distributions. We consider a data set, which represents the life-time data of relief times (minutes) of 20 patients receiving an analgesic reported by Gross and Clark (1975). The data set is given as follows:

<table>
<thead>
<tr>
<th>1.1</th>
<th>1.4</th>
<th>1.3</th>
<th>1.7</th>
<th>1.9</th>
<th>1.8</th>
<th>1.6</th>
<th>2.2</th>
<th>1.7</th>
<th>2.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>1.8</td>
<td>1.5</td>
<td>1.2</td>
<td>1.4</td>
<td>3.0</td>
<td>1.7</td>
<td>2.3</td>
<td>1.6</td>
<td>2.0</td>
</tr>
</tbody>
</table>

In order to compare the distributions, we consider the criteria like Bayesian information criterion (BIC), Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICC) and -2 logL. The better distribution is which corresponds to lesser values of AIC, BIC, AICC and – 2 log L. For calculating AIC, BIC, AICC and -2 logL can be evaluated by using the formulas as follows

\[ AIC = 2K - 2\log L, \quad BIC = k \log n - 2\log L, \quad AICC = AIC + \frac{2k(k+1)}{(n-k-1)} \]

Where \( k \) is the number of parameters, \( n \) is the sample size and -2 logL is the maximized value of log likelihood function.

Table. 1: MLE’s, S.E, -2 logL, AIC, BIC and AICC of the fitted distribution of the given data.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>MLE</th>
<th>S.E</th>
<th>-2 logL</th>
<th>AIC</th>
<th>BIC</th>
<th>AICC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aradhana</td>
<td>( \hat{\theta} = 1.1231938 )</td>
<td>( \hat{\theta} = 0.1526142 )</td>
<td>56.37</td>
<td>58.37</td>
<td>59.365</td>
<td>58.6</td>
</tr>
<tr>
<td>Weighted Aradhana</td>
<td>( \hat{c} = 7.984101 ) ( \hat{\theta} = 5.514582 )</td>
<td>( \hat{c} = 2.957678 ) ( \hat{\theta} = 1.610589 )</td>
<td>11.81</td>
<td>15.81</td>
<td>17.80</td>
<td>16.48</td>
</tr>
</tbody>
</table>

It can be easily seen that the weighted Aradhana distribution have the lesser AIC, BIC, AICC and -2 logL values as compared to Aradhana distribution. Hence, we can conclude that the weighted Aradhana distribution leads to better fit than the Aradhana distribution.
11. Conclusion

In the present study, we have introduced a new generalization of the Aradhana distribution termed as Weighted Aradhana distribution and has two parameters. The subject distribution is generated by using the weighting technique and the parameters have been obtained by using maximum likelihood technique. Some mathematical properties along with reliability measures are discussed. The new distribution with its applications in real lifetime data has been demonstrated. The results are compared with Aradhana distribution and has been found that weighted Aradhana distribution provides better fit than Aradhana distribution.

Reference


