UNDERSTANDING THE ADVANTAGES AND LIMITATIONS OF VISUALIZATION IN CALCULUS

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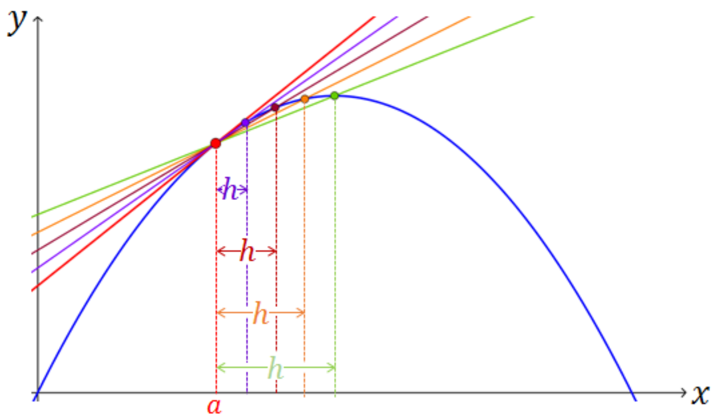
ABSTRACT :

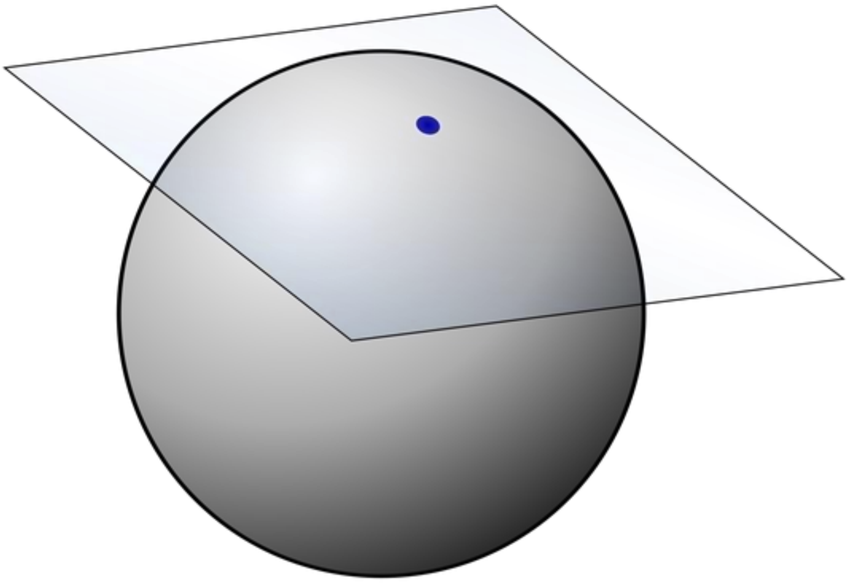
When intuition in reflected in context with calculus concepts, the visuals that come to mind are typically graphical. Graph representations are commonplace in calculus, but can they help to anchor intuitions of the big ideas. The drawback with graphs is that they are themselves just another representation for a relation (others might be symbolic or tabular for instance) and so are kind of already built upon a particular kind of abstraction. When dealing with concrete problems, some students might have trouble translating the problem into a relation (in order to do symbolic manipulation on it) then the intuition from graphs might not transfer as readily.

The two cases in differential calculus where graphs are useful for intuition is in functions of two and three variables. One end up with the notion of a tangent line as a limit of secant lines:

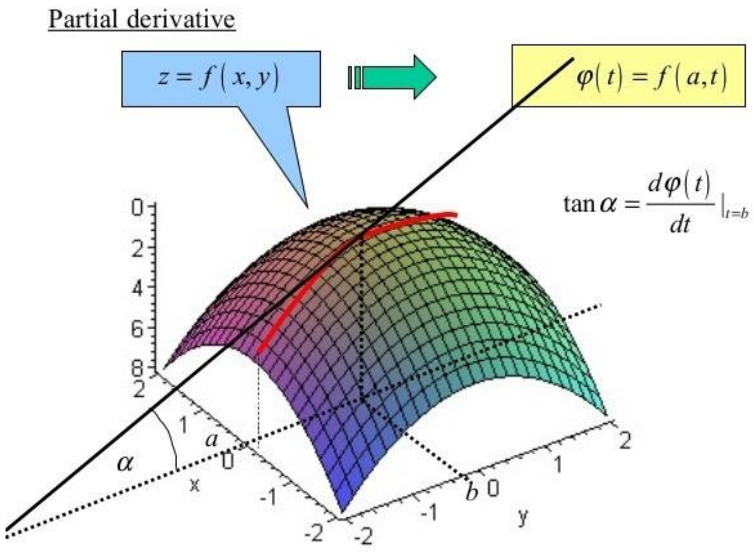


The tangent line touches the curve at one point and represents the precise slope of the curve at precisely that point. Imagine something small like a marble moving along a track (the curve) in space, so there is no gravity. The only thing that constrains the movement of the marble is the curve. If the track suddenly disappears, the marble would move off in a straight line in whatever direction it was headed at that precise moment. This is the tangent line.

In 3 dimensions you get a tangent plane as a limit of secant planes (here we illustrate on example for an interactive visualization of secant planes):

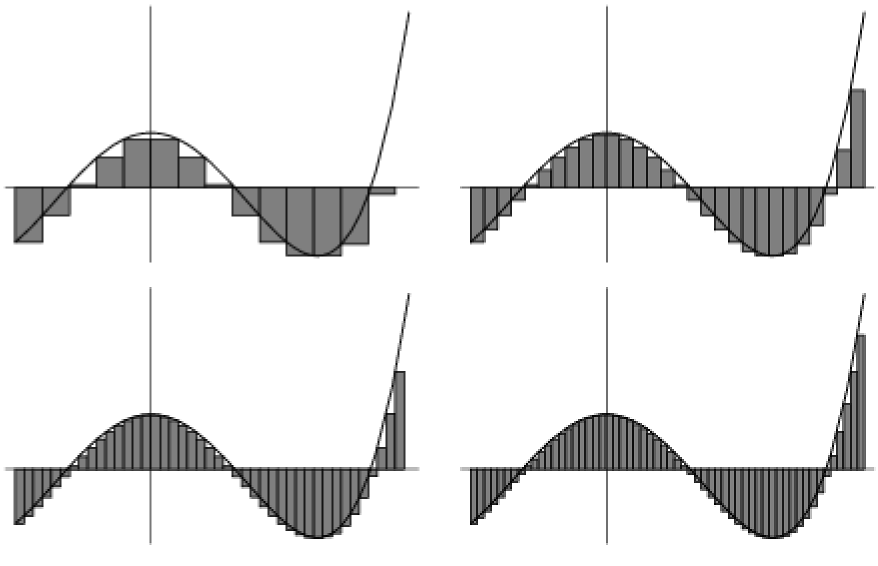


This plane touches the surface at just one point. The plane is formed by the partial derivatives of the function that defines the surface. The point of contact between the surface and the plane is infinitesimally small - practically non-existent. Now, imagine a marble (actually just a dot, but marbles make this more concrete) moving along the surface in one direction. If the surface suddenly disappears, it would move along a tangent **line**, the partial derivative (One can ignore pretty much everything on this diagram but the surface and tangent line):

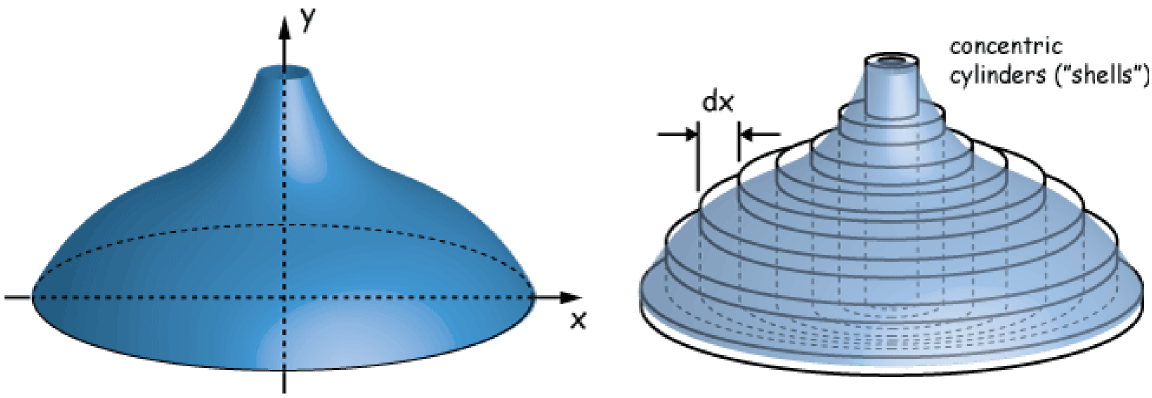


If that line spin around the point on the surface, perpendicular to the surface, you get the tangent plane.

For integral calculus, the same kind of thing is at play. The integral is the limit of smaller and smaller strips that approximate the area under a smooth curve:



(Note: technically, the area is between the curve and the axis. If this area falls above the axis, it's positive. If it falls below the axis, it's the negative of the area). The same applies in three dimensions here too. The volume is usually seen as the limit of taking smaller and smaller discs or columns or some other 3D shape whose volume is easy to calculate:



The key between both of these domains is limits. Students should learn how to calculate a derivative symbolically using limits. Then they typically derive some handy rules for limits and leave the definition behind, until their present tools reach their limit and a new rule is required. Then they will be required to revisit the definition with limits again. Limits are one of the big ideas students should always be mindful of. Continuity is a key property dealt with in calculus. The fundamental theorem of calculus, which conceptually integrates the derivative and the integral is also pretty big. However, since math is conceptually deep, in many cases it would be beneficial to remember every theorem introduced in the text or lecture in the order presented. Often times, theorem's build on each other conceptually. The abstraction of using a derived rule instead of the definition makes it easier to do calculations and solve more complex problems.

BACKGROUND :

Calculus, known historically as infinitesimal calculus, constitutes a major part of modern mathematics education. Its major branches, differential calculus and integral calculus, are related bythe fundamental theorem of calculus. Calculus is the study of change, in the same way that geometry is the study of shape and algebra is the study of operations and their application to solving equations. Yet the three domains of mathematics share concepts and generate new paths of development. If Cartesian geometry blended geometry with algebra, the history of perceivable linkages of calculus with both algebra and geometry are still longer. In fact geometry is the first candidate for providing the breakthrough in the emergence of the newer subject like calculus.

Calculus has widespread applications in science, economics, and engineering and can solve many problems for which algebraor geometry alone is insufficient. Calculating volumes and areas, the basic function of integral calculus. Egyptian mathematician successfully calculated the volume of a pyramidal frustum.Greek geometers are credited with a significant use of infinitesimals. Democritus is the first person recorded toconsider seriously the division of objects into an infinite number of cross-sections, but his inability to rationalize discrete cross-sections with a cone's smooth slope prevented him from accepting the idea. At approximately the same time, Zeno of Elea discredited infinitesimals further by his articulation of the paradoxes which they create. Antiphon and later Eudoxus are generally credited with implementing the method of exhaustion, which made it possible to compute the area and volume of regions and solidsby breaking them up into an infinite number of recognizable shapes.The Greek mathematician Archimedes was the first to find the tangent to a curve, other than a circle, in a method akin to differential calculus. While studying the spiral, he separated a point's motion into two components, one radial motion component and one circular motion component, and then continued to add the two component motions together thereby finding the tangent to the curve. The Indian mathematician-astronomer Aryabhata used a notion of infinitesimals and expressed an astronomical problem in the form of a basic differential equation.Manjula, in the 10th century, elaborated on this differential equation in acommentary. This equation eventually led Bhāskara II in the 12th century to develop the concept of a derivative representing infinitesimal change, and he described an early form of "Rolle's theorem".Newton and Leibniz are usually credited with the invention ofmodern infinitesimal calculus in the late 17th century. Their most important contributions were the development of the fundamental theorem of calculus. Also, Leibniz did a great deal of work with developing consistent and useful notation andconcepts. Newton was the first to organize the field into one consistent subject, and also provided some of the first and most important applications, especially of integral calculus. so we can see calculus emerges in two ways, one based on geometrical aspect and the other algebraic aspect (i.e logical symbolism).Geometry deals with figures and figures can be easily visualized .Therefore visualization can play important part for understanding of concepts of calculus.

VISUALIZATION AND CALCULUS :

This Paper describes some important aspects concerning the role of visualization in calculus learning. We consider an example from integral calculus which focuses on visual interpretations. The empirical study is based on four problems related to the integral concept that highlight various facets of visualization. In particular, our study is interested in the visual images that students use for working on specific problems and how they deal with given visualizations. The findings show the importance as well as the difficulties of visualization for the students.

The first part of the title is borrowed from a common understanding which highlights the importance of visualization in general. Likewise, visualization has a long tradition in mathematics and the list of famous mathematicians using or explicitly advocating visualization is large.One prominent example is certainly the blind Euler whose restriction did not have an effect on his creative power. During visual impairment the years of his blindness he was able to produce more than 355 papers – due to his visual imagination as well as his phenomenal memory (Draaisma, 2000)[1]. Of course in his unique case visualization was in the form of mental images of physical ones. Hadamard (1954)[2] pointed out the importance of visualization by referring to Einstein and Poincaré. They both emphasized using visual intuition. In Pólya's (1973) list of heuristic strategies for successful problem solving, one prominent suggestion is ”draw-a-figure” which has become a classic pedagogical advice. However, this Paper discusses some findings which focus on the role of visualization ranging from being useful to being an impediment.

VISUALIZATION IN MATHEMATICS LEARNING :

The role of visualization in mathematics learning has been the subject of much research (e.g. Arcavi, 2003[3]; Bishop, 1989[4]; Eisenberg & Dreyfus, 1986[5]; Presmeg, 1992[6];Stylianou & Silver, 2004[7]). In accordance to Zimmermann and Cunningham (1991)[8] as well as Hershkowitz et al.(1989), Arcavi (2003) defines visualization as follows:

Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

This definition emphasizes that, in mathematics learning, visualization can be a powerful tool to explore mathematical problems and to give meaning to mathematical concepts and the relationship between them. Visualization allows for reducing complexity when dealing with a multitude of information. However, the limitations and difficulties around visualization and even the reluctance to visualize have also been discussed (Arcavi, 2003; Eisenberg, 1994[9]; Stylianou & Silver, 2004). Visual techniques which rely on “not always procedurally ‘safe’ routines” (Arcavi, 2003) are considered to be cognitively more demanding than analytical techniques. In a different context, visualization is discussed as an important part of so-called “concept images” (Tall & Vinner, 1981[10]). The concept image includes visual images, properties and experiences concerning a particular mathematical concept. To understand a formal mathematical concept often the learner is required to generate a concept image for it. Nevertheless, Vinner (1997[11]) points out that “in some cases the intuitive mode of thinking just misleads us.” The Paper focus largely on the visual aspects of the concept image.

TREATMENT OF INTEGRAL CALCULUS IN SCHOOL :

Many topics in mathematics have large scope for visual interpretations and the integral calculus is certainly one of those. A classical approach to the integral in school is the area calculation problem. This problem allows for using the geometric reference for visualization. Thus, the most basic way of introducing integrals is using the close connection between the idea of an integral and the idea of an area, initially for functions with positive areas in the first quadrant. Later on, this idea is expanded by identifying the integral as sum of the oriented areas.

RESEARCH QUESTIONS :

Much of the research into mathematics students’ knowledge of the integral has been oriented by assumptions about what students should know. Instead, we report on some ongoing research into what students do know with a special focus on visual aspects of the integral. This Paper presents some results gained within the scope of a larger study to investigate students’ mental representations concerning the integral. Our research questions in this study were:

• What visual images do students have concerning the integral?

• How do students deal with a given visualization?

• To what extent are visual images used by the students?

METHODOLOGY :

The study employed qualitative methods to capture the importance of visualization in the learning of integral calculus. The observation of the lessons in question and the analysis of the teaching material led to constructing a questionnaire containing several problems related to the integral. The students worked on this questionnaire in the classroom under supervision and were allowed to use a calculator. For the purposes of this Paper we focused on four problems revealing diverse aspects of the integral. The subjects in this study were students in grade 12 of two Indian high-schools (East & West High School,Bela(Bihta,Patna) and Infant Jesus school Patna). The first class consisted of 20 students, 10 female and 10 male students. The second class consisted of 30 students, 8 female and 22 male students. The two classes together form a total of 50 students. For the analysis, we do not distinguish between these two classes.

EMPIRICAL RESULTS :

This is not the place to give a detailed analysis of the observed lessons. The main approach to the integral discussed above emerged in both classes. In this section, we restrict ourselves to the presentation of the problems, the underlying mathematical aspects and the students' answers.

Problem 1: *Draw a figure to illustrate the geometric definition of the integral.* **(**The geometric definition refers to the area concept as already mentioned. We were interested in the visual representations that students associate with this aspect of the integral.)

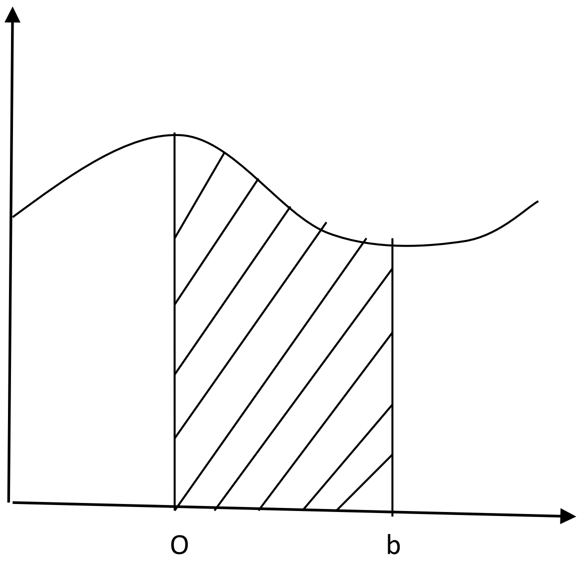


Figure 1

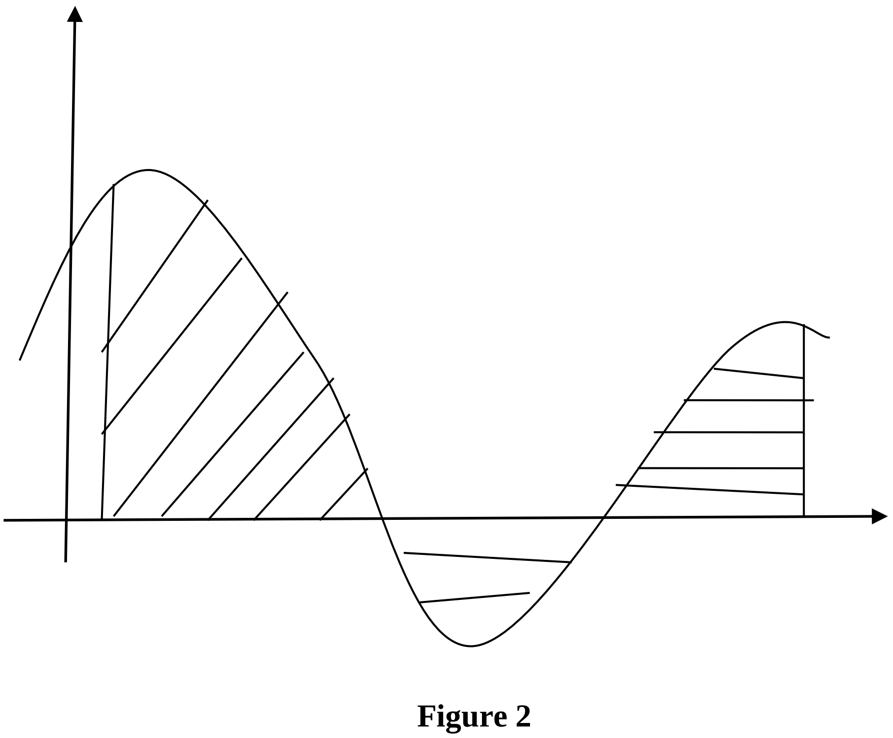


Figure 2

The following table shows the distribution of the students' solutions:

Table 1: Students' answers to problem 1

Positive area	Positive and negative areas	No answer
70%	20%	10%

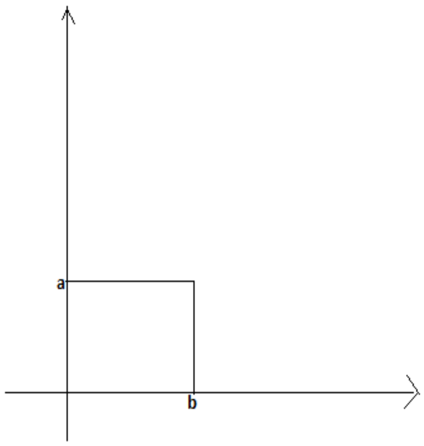
90% of the students were able to illustrate the geometric definition of the integral. However, it is remarkable that 70% of the students disposed of an image that is limited to a positive area. Figure 1 shows an example of such visualization which represents merely one aspect of the integral concept. This restricted visualization will turn out to be an
obstacle for working on the other problems.

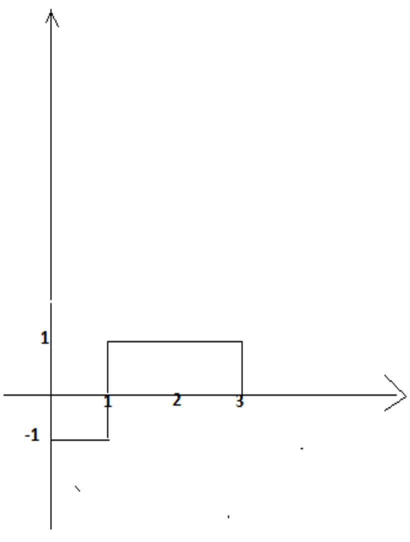
Figure 2 shows an example for a more adequate visualization which was only used by 20% of the students.

Problem 2:

Find a formula for the area by using integration.

a) b)





In contrast to problem 1, the students were given a concrete visualization and were asked to find the integrand as well as the limits of integration. In problem 2b, the students additionally had to consider the orientation of the area.

Table 2 shows the distribution of the students' answers to problem 2a:

Table 2: Students' answers to problem 2a

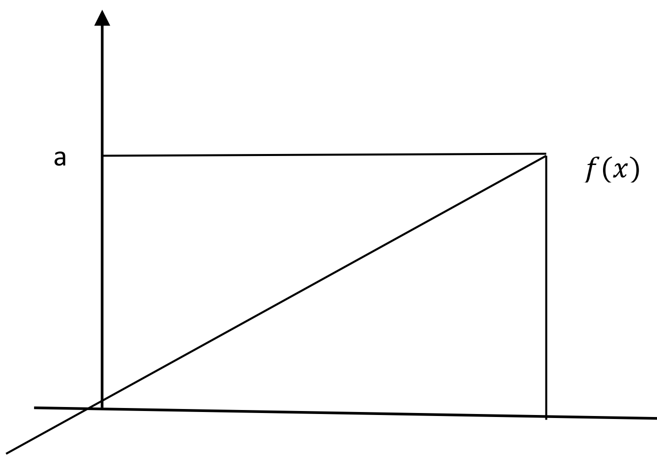
Correct answer	Incorrect answer	No answer
60%	30%	10%

It is notable that the given visualization of this problem differed only slightly from the visualization the students chose in problem 1. However, half of the students were not able to give a correct answer. Among the incorrect answers, the following terms can be found:

$$\int_a^b k\,dx \int_a^b a\,dx \int_a^b a(x)dx \;\; \int_a^b a\,da \int_a^b f(a)dx$$

One difficulty for the students was to name the limits of integration. It is evident that finding the integral for the given image conflicts with the standard notation: $\int_a^b f(x)dx$

Furthermore, the students had major problems to recognize the given constant function as a possible integrand. Obviously, they were missing an x-term. One student gave the correct answer but stated the following: $\int_0^b f(x)dx$: Not possible, because this is a constant function and there is no x in it and that's why it is not possible to put in the limits.
Two students solved this conflict by drawing a supporting straight line as shown in figure 3 .They obtained the answer to this problem in creative though complicated way.



$$y = \frac{a}{b}x + b$$

Where $x = 0, y = 0$

$\therefore b = 0$

$$\therefore \mathcal{A} = 2\int_o^b f(x)bx$$

$$= 2\int_o^b \frac{a}{b}x$$

Figure 3

Table 3 shows the distribution of the students' answers to problem 2b:

Table 3: Students' answers to problem 2b

Correct answer	Incorrect answer	No answer
50%	45%	5%

While the difficulties to find the integrand remained, the problem to name the limits of integration minimized due to the concrete numbers provided in the illustration. However ,a new obstacle emerged because of the orientation of the area. Instead of the area, the students calculated the integral. This was found in more than half of the incorrect answers. Some students solved this conflict by shifting the square above the x-axis.

Problem 3:

(a) Find the area bounded between the function f(x)= sinx and x-axis over [π,2π].

(b) Calculate the integral: $\int_{-\pi}^{2\pi} sinxdx$

For the answer to problem 3a the students had to calculate an area of negative orientation while in problem 3b the same function was given but this time they were asked to calculate the integral. Even though the limits of integration changed, the answer to problem 3b could be immediately given by visualizing the graph of the function and considering problem 3a. This problem demanded that students distinguish clearly between the area and the integral concept.

Table 4 shows the distribution of the students' answers to problem 3a:

Table 4: Students' answers to problem 3a

Correct answer	Incorrect answer	No answer
30%	65%	5%

Some of the students had difficulties to put in the limits or to give the correct antiderivative. More interestingly, 65% of the incorrect answers resulted in giving a negative value as area of the function. On the one hand, these students did not use visualization to approach the problem. On the other hand, they did not scrutinize the negative value of their result. Only 8% of all students sketched the graph.

Table 5 shows the distribution of the students' answers to problem 3b:

Table 5: Students' answers to problem 3b

Correct answer	Incorrect answer	No answer
25%	70%	5%

The distribution of the answers to problem 3b is similar to 3a and the same mistakes emerged. Remarkably, in 47% of the incorrect answers a positive value was given. The students continued calculating the area as required in 3a instead of the integral, some of them even mentioned explicitly “A=2”. Only 8% of all students visualized the graph, 6% referred to their solution of problem 3a.

Problem 4

How would you proceed to calculate $\int_{-1}^{1} sinx^5 dx$

This problem can be easily solved by visualization. The function is odd so that on the given interval [-1,1] the integral equals zero. Only 6% of all students took into account these considerations. 8% of the students did not answer at all while the other students proposed to work out the integral by substitution (40%), by finding the antiderivative (29%) or by integration by parts (17%). To summarize, the solutions to this task showed an explicit bias towards an algorithmic approach even though the visual one would have been significantly easier.

CONCLUSIONS

The selected problems emphasize convincingly some important aspects inherent to visualization. On the one hand, visualization proves to be a useful tool for working on the problems and the common understanding mentioned in the title seems to be appropriate. For example, some students use visualization in a creative way by modifying the given task (problem 2; figure 3). This approach enables them to avoid the difficulties with the given visualization and thereby sheds light on the underlying obstacles concerning this task. Another interesting point is that even students that do not show visualization on their paper were able to solve problem 3 correctly. This highlights once again the importance of pictures in the mind (Presmeg, 1986[12]). The students in this study largely demonstrated their ability in visualizing the geometric definition of the integral (problem 1; figure 1). Nevertheless, their chosen visualization only reflects one particular aspect of the integral concept. This entails some important consequences for working on the other problems. For example, the restricted visualization proves to be a hindrance for the solution of problem 3. The connection of the integral with the area misleads the students not to distinguish clearly between the two concepts. Basically, both concepts are different from each other, but at the same time, they have a certain though marginal intersection which predominates the students' thinking.

Students use visualization to solve the problems 2 and 3, this does not mean that they are able to solve the problems correctly. They do not dispose of the cognitive flexibility to use both visual and algorithmic techniques (Arcavi, 2003). The students usually did not choose to visualize in problem 4 but proposed an algorithmic approach instead. They are cognitively fixed on algorithms and procedures instead of recognizing the advantages of visualizing this problem – a phenomenon which Eisenberg (1994) describes as reluctance to visualize. The visualization given in problem 2 differs only slightly from the visualizations the students gave in problem 1. However, most of the students were not able to deal with this given visualization and to adequately interpret the information given in this problem. These aspects highlight the ambivalence of visualization as Tall (1994[13]) points out: "It is this quality of using images *without being enslaved by them* which gives the professional mathematician an advantage but can cause so much difficulty for the learner". Hence, the importance of visualization for mathematics learning and teaching is constituted in being aware of the fact that visualization never represents an isomorphism of mathematical concepts and their relationships. Therefore, visualization should be accompanied by reflective thinking to avoid being enslaved by it.

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