New Results on Countably Barrelled Spaces and Countably Infrabarrelled Spaces

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Introduction

In this paper we define inductive limit and strict inductive limit. Then find the relation between inductive limit and countably Barrelled or countably infrabarrelled spaces. Also we have to find the separate relation between countably Barrelled spaces and countably infrabarrelled spaces.

Key words: Countably Barrelled, Countably infrabarrelled, inductive limit, strict inductive limit.

Some Basic Definition :

(i) Countably Barrelled – A locally convex space E is said to be countably barrelled, if every σ-barrel is a neighbourhood of zero.

(ii) Countably Infrabarrelled – A locally convex space E is said to be countably infrabarrelled, if every bornivorous σ-barrel in E is neighbourhood of zero.

(iii) Inductive limit – Let $\{x_i\}_{i \in I}$ be a family of locally convex spaces and for each $i \in I$ let f_i be a linear mapping of x_i into a vector space *x* such that $\bigcup_{i \in I} f_i(x_i)$ spans *x*. Then there is finest locally convex topology on *x* under which all the mapping f_i are continuous. The locally convex space *x* with this topology is called the inductive limit of the locally convex spaces x_i by mappings f_i.

(iv) Strict inductive limit – Let E be a vector space and let $(E_n)_{n \in N}$ be a sequence of linear subspaces of E such that $E_n \subset E_{n+1}$ for all $n \in N$ and $E = \bigcup_{n \in N} E_n$. Suppose that each E_n is equipped with a locally convex topology τ_n and for each *n*, the topology induced by τ_{n+1} on E_n is τ_n. Let τ be the finest locally convex topology on E for which all the canonical one-one mapping $f_n : E_n \to E$ are continuous. Then τ induces on E_n the topology τ_n. E is said to be the strict inductive limit of the sequence (E_n).

NEW RESULTS

Result (1) : Let E be a locally convex space with E has a Mackey topology $\tau(E, E')$, suppose $E = \bigcup_1^{\infty} E_n$, where $\{E_n\}$ is an expanding sequence of subspace of E. It $E' = \bigcap_1^{\infty} E'_n$ then E is the strict inductive limit of the sequence $\{E_n\}$.

Proof : Let F (ξ) be any locally convex space, and $t : E \to F$ a linear mapping whose restriction t_n to E_n is continuous. If we show that *t* is continuous, then the result is established.

Let $f \in F'$. Then composite mapping $fot_n : E_n \to K$ (scalars) is continuous i.e. $fot_n \in E'_n$ for each n. Hence $fot \in E'$, so *t* is σ (E, E') - σ (F, F') continuous. Hence *t* is $\tau(E, E') - \tau(F, F')$ continuous, hence $\tau - \xi$ continuous.

Result (2) : Let E be a locally convex space with topology $\tau \neq \tau(E, E')$, with $E = \bigcap_1^{\infty} E_n$ where $\{E_n)$ is an expanding sequence of subspaces. Then E is the inductive limit of $\{E_n\}$ in either of the following cases.

(i) E is countably barrelled.

(ii) E is countably infrabarrelled, and every bounded subset of E is contained in some E_n.

Proof : $E' = \bigcap_1^{\infty} E'_n$ if either (i) $E'\left[\sigma(E', E)\right]$ is sequentially complete or (ii) $E'\left[\beta(E', E)\right]$ is sequentially complete, and every bounded subset of E is contained in some E_n. In fact for countable barrelled spaces the result follows even if the topology is not the Mackey topology.

Result (3) : Let E be a countably barrelled or countably infrabarrelled space, and F a closed subspace of E of countable co-dimension, and such that for every bounded subset B of E, F is of finite codimension in span $\{F \cup B\}$. Then F is countably barrelled or countably infrabarrelled.

Proof : Let $\{x_n\}$ be a sequence in E forming a base for a complentary subspace G of F. Put $E_1 = F, E_n = span\left(E_{n-1}, x_{n-1}\right)(n > 1)$. Then $E = \bigcup_1^\infty E_n$ and E is a strict inductive limit of the sequence $\{E_n\}$. Since F is closed, each E_n is closed. Consider the projection map $n : E \to F$, parallel to G. The restriction $\pi_n : E_n \to F$ is continuous, since F is closed and of finite condimension in E_n. Since E is the inductive limit of the sequence $\{E_n\}, \pi$ is continuous.

It follows that F has a closed complement in E, and that F is isomorphic with a quotient of E by a closed subspace. Since property of being countably barrelled or countably infrabarrlled is preserved, when passing to quotients, F is countably barrelled or countably infrabarrelled.

Result (4) : Let E be a locally convex space such that $E'\left[\sigma\left(E', E\right)\right]$ is sequentially complete. If A is a closed absolutely convex subset of E such that span A is a contable co-dimension in E, then span A is closed.

Proof : Let $E = span\, A \oplus span\,\{x_n\}$ where $\{x_n\}$ is a linearly independent sequence. We construct $g_k \in E'$ such that $g_k\left(x_i\right) = \delta_{k_i}$ and $g_k(a) = 0$ for each $a \in A.$

Let $B_r = T(A, x_i \ldots\ldots\ldots\ldots, x_{k-1}, x_k, x_{k+1}, \ldots\ldots x_r\}(r > K).$ Then B_r is absolutely convex and closed, and $x_k \notin rB_r.$ By Hahn – Banach theorem, there exist $f_r \in E'$ such that $f_r\left(x_n\right) = 1$ and $f_r(x) \le 1/r$ for each $x \in B_r.$ The sequence $\{f_r\}_{r > k}$ is a Cauchy sequence in $\sigma\left(E', E\right)$ and hence converges to some $g_k \in E'$. Since span $A = \bigcap_1^\infty g_k^{-1}(0)$, span A is closed.

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