A new oscillation criteria for first order nonlinear differential equation with non-monotone advanced arguments

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Abstract

In this paper, we present a new sufficient condition for the oscillation of first order nonlinear advanced differential equation with deviating arguments involving limit supremum and limit infemum using Grönwall inequality. An illustrative example related to our result is given.

Keywords: Advanced differential equation, non-monotone argument, oscillatory solutions, Grönwall inequality.

1. Introduction

In this paper, we shall study the oscillatory behavior of the first order nonlinear advanced differential equation

$$u'(t) - a(t)g(u(\delta(t))) = 0 \qquad (1.1)$$

where $t \geq t_0 > 0$.

Througout this paper we assume the following assumptions hold :

(H_1) $a(t), \delta(t) \in \square([t_{0,}\infty), R)$, $\delta(t)$ is non-monotone or nondecreasing and $a(t) > 0$.

(H_2) $\delta(t) \geq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \delta(t) = \infty$.

(H$_3$) $g \in \square(R, R)$ and $u\delta(u) > 0$ for $u \neq 0$.

By a solution of (1.1), we mean a continuously differentiable function u(t) defined on $[t_0, \infty)$ and satisfying (1.1) for all $t \geq t_0$. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *non oscillatory*.

Oscillatory behavior of advanced differential equations with deviating arguments have been extensively studied by many authors, see for example [1] to [10] and references therein.

In [7] the authors obtained that if

$$\liminf_{t\to\infty} \int_t^{t+T} p(s)ds > \frac{1}{e}$$

then all solutions of the equation

$$u'(t) - p(t)u(t+T) = 0 \ , \qquad t \geq t_0 \qquad (1.2)$$

are oscillatory.

In particular if $P(t) \equiv p \in (0, \infty)$ they also proved that the condition

$$PT > \frac{1}{e}$$

is necessary and sufficient for

a) The differential inequality of advanced type $u'(t) - p(t)u(t+T) \geq 0$, $t \geq t_0$ has no eventually positive solution.

b) The differential inequality of advanced type $u'(t) - p(t)u(t+T) \leq 0$, $t \geq t_0$ has no eventually negative solution.

c) All solutions of (1.2) are oscillatory.

In [3] the authors considered the non linear differential equation

$$u'(t) + p(t)g(u(\tau(t)) = 0 \qquad t \geq t_0, \qquad (1.3)$$

where $p(t) \leq 0$, $\tau(t) \geq t$ is non decreasing for $t \geq t_0$ and proved that (1.3) is oscillatory if

$$\liminf_{t\to\infty} \int_t^{\tau(t)} (-p(s))ds > \frac{M}{e},$$

where $M = \limsup_{|t|\to\infty} \frac{|t|}{|f(t)|} < \infty$.

In [9] the author studied the oscillatory behavior of advanced differential equations with deviating arguments.

2. Oscillation results

In this section, we shall establish some oscillation criteria for the equation (1.1) under the assumption that $\delta(t)$ is non-monotone. Set

$$h(t) := \inf_{s\geq t} \delta(s), \qquad t \geq 0 \qquad (2.1)$$

Clearly, $h(t)$ is non decreasing and $\delta(t) \geq h(t)$ for all $t \geq 0$.

Assume that the function g in (1.1) satisfies the following condition

$$\limsup_{|u|\to\infty} \frac{u}{g(u)} = L, \qquad 0 \le L < \infty. \qquad (2.2)$$

Lemma2.1(Grönwall inequality)

If

$$u'(t) - a(t)u(t) \ge 0, \qquad t \ge t_0, \qquad (2.3)$$

where $a(t) \ge 0$ and $u(t) \ge 0$, then we have

$$u(s) \ge u(t)\exp\{\int_t^s a(u)du\}, \qquad s \ge t \ge t_0. \qquad (2.4)$$

Lemma 2.2[4]

Assume that (1.1) holds and

$$\liminf_{t\to\infty} \int_t^{\delta(t)} a(s)ds = m > 0,$$

then we have

$$\liminf_{t\to\infty} \int_t^{\delta(t)} a(s)ds = \liminf_{t\to\infty} \int_t^{h(t)} a(s)ds = m, \qquad (2.5)$$

where h(t) is defined in (2.1).

Proof:

Let $\delta(t), h(t)$ be non decreasing and let $t < h(t) \le \delta(t)$ for all $t \ge 0$. Then

$$\int_t^{h(t)} a(s)ds \le \int_t^{\delta(t)} a(s)ds$$

and consequently

$$\liminf_{t\to\infty} \int_t^{h(t)} a(s)ds \le \liminf_{t\to\infty} \int_t^{\delta(t)} a(s)ds.$$

If (2.5) does not hold, then there exists a sequences $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and $0 < m_1 < m$ such that

$$\lim_{k\to\infty} \int_{t_k}^{h(t_k)} a(s)ds \le m_1 < m.$$

Since $h(t_k) := \inf_{s \ge t_k} \delta(s)$, there exists a $t_k^{'} \ge t_k$ such that $h(t_k) = \delta(t_k^{'})$. Therefore we have

$$\int_{t_k}^{h(t_k)} a(s)ds = \int_{t_k}^{\delta(t_k^{'})} a(s)ds \geq \int_{t_k^{'}}^{\delta(t_k^{'})} a(s)ds .$$

It follows that $\{\int_{t_k^{'}}^{\delta(t_k^{'})} a(s)ds\}_{k=1}^{\infty}$ is a bounded sequence having a convergent subsequence $\left\{\int_{t_{n_k}'}^{\delta(t_{n_k}')} a(s)ds\right\}$ such that

$$\int_{t_{n_k}'}^{\delta(t_{n_k}')} a(s)ds = c \leq m_1,$$

which implies

$$\liminf_{t\to\infty} \int_{t}^{\delta(t)} a(s)ds = \lim_{k\to\infty} \int_{t_k}^{\delta(t_k)} a(s)ds .$$

$$= \lim_{k\to\infty} \int_{t_{n_k}'}^{\delta(t_{n_k}')} a(s)ds \leq m_1 < m$$

This contradicts (2.5).

Theorem 2.1

Assume that the assumptions (H_2), (H_3) and the condition (2.2) hold. If $\delta(t)$ is non-monotone or non decreasing and if

$$\liminf_{t\to\infty} \int_{t}^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\}ds > \frac{L}{e}, \tag{2.6}$$

where $\delta(t)$ and $h(t)$ are as defined by (2.1), then all solutions of (1.1) oscillate.

Proof:

Assume the contrary, that there exists a non oscillatory solution $u(t)$ of (1.1). Since $-u(t)$ is also a solution of (1.1) whenever $u(t)$ is a solution of (1.1), we can confine ourselves only to the case where the solution $u(t)$ of (1.1) is eventually positive. Then, there exists $t_1 > t_0$ such that $u(t) > 0$, $u(\delta(t)) > 0$ and $u(h(t)) > 0$ for all $t \geq t_1$.
from (1.1) it follows that

$$u'(t) \geq a(t)g(u(\delta(t))) \geq 0 \text{ for all } t \geq t_1 \tag{2.7}$$

and therefore $u(t)$ is non-decreasing .

2.1. The case $L > 0$

In view of (2.2) we can choose $t_2 > t_1$, so large such that

$$g(u(t)) \geq \frac{1}{2L} u(t) \text{ for all } t \geq t_2. \tag{2.8}$$

Again from (1.1), we have

$$\frac{u'(t)}{u(t)} - a(t)\frac{g(u(\delta(t)))}{u(t)} = 0 \text{ for all } t \geq t_2. \tag{2.9}$$

Now integrating (2.9) from t to $h(t)$ and using the monotonicity of $u(t)$, we get

$$ln\frac{u(h(t))}{u(t)} - \int_t^{h(t)} a(s)\frac{g(u(\delta(s)))}{u(s)} ds = 0 \text{ for all } t \geq t_2. \tag{2.10}$$

now using (2.8) in (2.10) we get

$$ln\frac{u(h(t))}{u(t)} - \frac{1}{2L}\int_t^{h(t)} a(s)\frac{u(\delta(s))}{u(s)} ds \geq 0 \text{ for all } t \geq t_2. \tag{2.11}$$

By Grönwall inequality , we have

$$u(\delta(s)) \geq u(h(t))\exp\{\int_{h(t)}^{\delta(s)} a(u)du\} \text{ for all } t \geq t_2. \tag{2.12}$$

Using (2.12) in (2.11), we obtain

$$ln\frac{u(h(t))}{u(t)} - \frac{1}{2L}\int_t^{h(t)} a(s)\frac{u(h(t))}{u(s)}\exp\{\int_{h(t)}^{\delta(s)} a(u)du\} ds \geq 0, \tag{2.13}$$

using $t \leq s \leq h(t) \leq \delta(t)$ and the monotonicity of $u(t)$ we have $\frac{u(h(t))}{u(s)} \geq 1$. Therefore (2.13) becomes

$$ln\frac{u(h(t))}{u(t)} - \frac{1}{2L}\int_t^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\} ds \geq 0. \tag{2.14}$$

Now from (2.6), there exists $d > \frac{1}{4e}$ such that

$$\int_t^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\} ds = 8Ld > \frac{L}{e} \text{ for all } t \geq t_2,$$

Therefore

$$\frac{1}{2L}\int_t^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\} ds = 4d \ . \tag{2.15}$$

Now combining (2.14) and (2.15) we get

$$ln\frac{u(h(t))}{u(t)} - 4d \geq 0 \text{ for all } t \geq t_3 > t_2,$$

That is

$$\frac{u(h(t))}{u(t)} \geq e^{4d} \geq 4ed > 1 \ . \tag{2.16}$$

Repeating the above procedure *k* times we get

$$\frac{u(h(t))}{u(t)} \geq (4ed)^k \to \infty \text{ as } k \to \infty, \tag{2.17}$$

since $4ed > 1$.

Now from (2.6) there exists $t^* \in (t, h(t))$ such that

$$\int_t^{t^*} a(s) \exp\left(\int_{h(t^*)}^{\delta(s)} a(u)du \right) ds > \frac{L}{2e} \tag{2.18}$$

and

$$\int_{t^*}^{h(t)} a(s) \exp\left(\int_{h(t)}^{\delta(s)} a(u)du \right) ds > \frac{L}{2e}. \tag{2.19}$$

By Lemma (2.2), we have

$$\liminf_{t\to\infty} \int_t^{\delta(t)} a(s)ds = \liminf_{t\to\infty} \int_t^{h(t)} a(s)ds,$$

where $h(t) = \inf \delta(s)$.

By (1.1), we have

$$u'(t) \geq a(t)g(u(h(t))) \geq 0 \text{ for } t \geq t_3. \tag{2.20}$$

Integrating the last inequality from t to t^*, we get

$$u(t^*) - u(t) \geq \int_t^{t^*} a(s)g(u(h(s)))ds$$

or

$$u(t^*) \geq \int_t^{t^*} a(s)g(u(h(s)))ds$$

Using (2.8) in the last inequality we get

$$u(t^*) \geq \frac{1}{2L}\int_t^{t^*} a(s)u(h(s))ds$$

Now using Grönwall inequality we obtain

$$u(t^*) \geq \frac{1}{2L} u(h(t)) \int_t^{t^*} a(s) \exp\left(\int_{h(t)}^{\delta(s)} a(u)du \right) ds,$$

Now using (2.18) the last inequality becomes

$$u(t^*) \geq \frac{u(h(t))}{4e} \text{ for all } t \geq t_3. \tag{2.21}$$

Similarly integrating (2.20) from $t*$ to $h(t)$ and also using (2.19), we obtain

$$u(h(t)) \geq \frac{u(h(t^*))}{4e} \text{ for all } t \geq t_3. \tag{2.22}$$

Combining (2.21) and (2.22), we get

$$u(t^*) \geq \frac{u(h(t))}{4e} \geq \frac{u(h(t^*))}{(4e)^2}.$$

That is

$$\frac{u(h(t^*))}{u(t^*)} \leq (4e)^2 < \infty.$$

which is a contradiction to (2.17).

2.2. The case $L = 0$

In this section, we assume that

$$\limsup_{|u|\to\infty} \frac{u(t)}{g(u(t))} = 0. \qquad (2.23)$$

Since $\frac{u(t)}{g(u(\delta(t)))} > 0$, there exists $\varepsilon > 0$ such that $\frac{u}{g(u)} < \varepsilon$ or $\frac{g(u)}{u} > \frac{1}{\varepsilon}$.

Now from (1.1) we have $u'(t) > \frac{u(\delta(t))}{\varepsilon} a(t)$ for all $t \geq t_0$.

(2.24)

Integrating (2.24) from t to $h(t)$, we get

$$u(h(t)) - u(t) > \frac{1}{\varepsilon} \int_t^{h(t)} a(s) u(\delta(s)) ds,$$

That is

$$u(h(t)) > \frac{1}{\varepsilon} \int_t^{h(t)} a(s) u(h(t)) \exp\left(\int_{h(t)}^{\delta(s)} a(u) du \right) ds$$

or

$$u(h(t)) > \frac{u(h(t))}{\varepsilon} \int_t^{h(t)} a(s) \exp\left(\int_{h(t)}^{\delta(s)} a(u) du \right) ds$$

or

$$1 > \frac{L}{e\varepsilon}$$

or

$$\varepsilon > \frac{L}{e}.$$

which is a contradiction to $\lim_{|u|\to\infty} \frac{u(t)}{g(u(t))} = 0$.

The proof is completed.

Theorem 2.2

Assume that the assumptions (H_2), (H_3) and the condition (2.2) hold and if

$$\limsup_{t\to\infty} \int_t^{h(t)} a(s) \exp\{ \int_{h(t)}^{\delta(s)} a(u) du \} ds > L, \qquad (2.25)$$

where $\delta(t)$ is non-monotone or nondecreasing and $h(t)$ is defined as in (2.1), then all the solutions of (1.1) oscillate.

Proof:

Assume for the sake of contradiction, that there exists a non oscillatory solution $u(t)$ of (1.1). Since $-u(t)$ is also a solution of (1.1), whenever $u(t)$ is a solution of (1.1) therefore it is enough to prove the theorem for positive solutions of (1.1). Then, there exists $t_1 \geq t_0$ such that $u(t) > 0, u(\delta(t)) > 0$ and $u(h(t)) > 0$ for all $t \geq t_1$.Then, from (1.1) we have

$$u'(t) = a(t)g(u(\delta(t))) \geq 0 \text{ for all } t \geq t_1.$$

and therefore $u(t)$ is non-decreasing for all $t \geq t_1$.

From (2.8) we obtain

$$u'(t) = a(t)g(u(\delta(t))) \geq \frac{1}{2L}a(t)u(\delta(t)) \text{ for all } t \geq t_1. \tag{2.26}$$

Integrating (2.26) from t to $h(t)$ and using the monotonicity of $u(t)$, we have

$$u(h(t)) - u(t) \geq \frac{1}{2L}\int_t^{h(t)} a(s)u(\delta(s))ds. \tag{2.27}$$

Using Lemma (2.2) in (2.27) and using Grönwall inequality, we have .

$$u(h(t)) \geq \frac{u(h(t))}{2L}\int_t^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\}ds \tag{2.28}$$

That is

$$u(h(t)\left(1 - \frac{1}{2L}\int_t^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\}ds\right) \geq 0,$$

or

$$\int_t^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\}ds \leq 2L.$$

Taking lim supremum, we have

$$\limsup_{t\to\infty}\int_t^{h(t)} a(s)\exp\left(\int_{h(t)}^{\delta(s)} a(u)du\right)ds \leq 2L \text{ for all } t \geq t_1. \tag{2.29}$$

But from (2.25) we have

$$\limsup_{t\to\infty}\int_t^{h(t)} a(s)\exp\left(\int_{h(t)}^{\delta(s)} a(u)du\right)ds = Q > L.$$

Then

$$L < \frac{Q+L}{2} < Q.$$

By choosing $2 = \frac{Q+L}{4L} > 1$ we have

$$\limsup_{t\to\infty}\int_{t}^{h(t)} a(s)\exp\left(\int_{h(t)}^{\delta(s)} a(u)du\right)ds = Q \le 2L = \frac{Q+L}{2}.$$

which is a contradiction to $\frac{Q+L}{2} < Q$.

3 Example

Example 3.1. Consider the equation

$$u'(t) - \frac{1}{e}u(\delta(t))In(10+|u(\delta(t))|) = 0, \quad t \ge 1,$$

where

$$\delta(t) = \begin{cases} 4t-6k-2, if\ t\in[2k+1, 2k+2] \\ -2t+6k+10, if\ t\in[2k+2, 2k+3] \end{cases} k \in \mathbb{N}_0. \tag{2.26}$$

By (2.1), we have

$$h(t) = \begin{cases} 4t-6k-2, & if\ t\in[2k+1, 2k+1.5] \\ 2k+4, & if\ t\in[2k+1.5, 2k+3] \end{cases} \qquad k \in \mathbb{N}_0. \tag{2.27}$$

Comparing the equation (2.26) with equation (1.1) we have $a(t) = \frac{1}{e}$ and $g(u) = uln(10+|u|)$, then we get

$$L = \limsup_{|u|\to\infty} \frac{u}{uln(10+|u|)} = 0.$$

Now, at $t = 2k+1.5$, $k \in \mathbb{N}_0$, we have

$$\int_{t}^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\}ds = \frac{1}{e}\int_{2k+1.5}^{2k+4} \exp\{\frac{1}{e}\int_{2k+4}^{4s-6k-2} du\}ds$$

$$= \frac{1}{e}\int_{2k+1.5}^{2k+4} \exp\{\frac{1}{e}(4s-8k-6)\}ds$$

$$= \frac{1}{4}\{\exp\frac{10}{e} - 1\} \approx 3.6418800876 > 1.$$

That is , we observe that

$$\liminf_{t\to\infty}\int_{t}^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\}ds > 1 > \frac{L}{e}$$

and

$$\limsup_{t\to\infty}\int_{t}^{h(t)} a(s)\exp\{\int_{h(t)}^{\delta(s)} a(u)du\}ds > 1 > L.$$

That is, all conditions of Theorem2.1 and Theorem2.2 are satisfied. Therefore all solutions of (1.1) oscillate.

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