SOME NEW RESULTS ON VARIATIONAL INEQUALITY PROBLEM IN HAUSDORFF TOPOLOGICAL VECTOR SPACE

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ABSTRACT

Variational inequality theory accomplished its present-day significance because of its various applications in the investigation of numerous true problems, specifically equilibrium problems. Equilibrium problems theory saw as hazardous development in applications over a wide class of direct and nonlinear problems being brought up in picture reproduction, clinical imaging, environment, network investigation, elasticity and nonlinear optimization. A classical speculation in this theory for equilibrium problems is convexity supposition and an equilibrium condition. Such sorts of problems are perceived as the classical equilibrium issue. The optimization problems, the variational inequality problems and the fixed point problems, considered as three standard instances of equilibrium problems. Consequently conjointly, equilibrium problems spread a broad scope of applications.

Keyword: *variational inequality, equilibrium, Hausdorff Topological, Vector space*

Introduction

Variational inequalities, presented by Hartman, Stampacchia and Browder, have been grown quickly for almost thirty years. Variational inequalities not just have animated new outcomes managing nonlinear partial differential conditions, yet in addition have been utilized in an enormous assortment of problems emerging in elasticity, basic examination, financial aspects, optimization, oceanography and provincial, physical and engineering sciences and so forth. This theory was grown at the same time not exclusively to consider the basic realities about the subjective conduct of arrangements of nonlinear problems, yet additionally to comprehend them all the more productively mathematically. Indeed, this theory gives us a sound premise to figuring the inexact arrangement of many moving and free limit esteem problems in a brought together structure.

VARIATIONAL INEQUALITIES

In this part, we present a concise presentation of different sort of variational inequality problems.

Numerous problems of elasticity and fluid mechanics can be communicated as far as an obscure u, speaking to the removal of a mechanical system, fulfilling

$$a(u,v-u) \geq F(v-u), \text{ for all } v \in K, \quad (1)$$

where the set K of allowable removals is a shut curved subset of a Hilbert space H, a(.,.) is a bilinear structure and F is a limited straight utilitarian on H. The relations of the sort (1) are called variationalinequalities. On the off chance that the bilinear structure a(.,.) is consistent, at that point by Riesz-portrayal Theorem 1.2.2, we have

$$a(u,v) = \langle Au, v \rangle, \text{ for all } u,v \in H, \quad (2)$$

where A will be a persistent linear operator on H. At that point the inequality (1.3.1) is identical to discover u€K with the end goal that

$$\langle Au, v-u \rangle \geq \langle F, v-u \rangle, \text{ for all } v \in K. \quad (3)$$

In the event that the operators An and F are nonlinear, at that point the variational inequality (3) is known as firmly nonlinear variational inequality, presented and concentrated by Noor . In the event that F = 0, at that point (3) is identical to discover u€K with the end goal that

$$\langle Au, v-u \rangle \geq 0, \text{ for all } v \in K. \quad (4)$$

The variational inequality of the sort (4) was presented and concentrated by Fichera in 1964. Lions and Stampacchia demonstrated the presence of remarkable arrangement of (4) utilizing basically the projection techniques. It merits referencing that the one-sided contact problems including contact laws of droning nature don't prompt the definition of variational disparities legitimately. Nonetheless, it has been appeared by Panagiotopoulus , utilizing the ideas of Clarke's summed up angle and Rockafeller's upper subderivative, that the nonconvex one-sided contact problems must be portrayed by a class of firmly nonlinear variational disparities .

Over the most recent twenty years, variational disparities have been summed up and stretched out in different ways. Variational-like inequality is one of its summed up structure which is presented and concentrated by Parida, Sahoo and Kumar

Leave K alone a closed curved set in R^n.. Given two nonstop guides $F:K \to R^n$ and $\eta:K \times K \to R^n$, then the variational-like inequality issue is to discover u€K with the end goal that

$$\langle F(u), \eta(u,v)\rangle \geq 0, \text{ for all } v \in K. \tag{5}$$

In the event that η(u,v) = V - u, at that point variational-like inequality (5) is comparable to the variational inequality (4). Leave X and Y alone two genuine Banach spaces. Let $K \subset X$ be a nonempty shut curved subset in $X, T: K \to L(X,Y),$ a mapping, where $L(X,Y)$ is the space of all direct nonstop operators from X to Y. Let $\{C(u) : u \in K\}$ be a group of shut pointed arched cone in Y with $\text{int}C(u) \neq \phi$ for each u€K, where intC(u) is the inside of the set C(u). At that point the issue of finding $u_0 \in K$ with the end goal that

$$\langle T(u_0), u - u_0\rangle \notin -\text{int}\,C(u_0), \text{ for all } u \in K, \tag{6}$$

is called vector variational inequality issue which is presented by Giannessi . Inequality (6) is called vector variational inequality.

A vector variational inequality in a finite-dimensional Euclidean space was first presented by Giannessi, which is the vector-valued rendition of the variational inequality of Hartman and Stampacchia . Later on, Chen et al., Konnov et al., Lee et al. , Lee et al. , Park et al. , Siddiqi et al., Yang, and Yu et al. contemplated a few sorts of vector variational imbalances in unique spaces. Ansari presented and considered vector variational-like disparities. From that point forward, Lee et al. what's more, Siddqi et al. have contemplated different vector variational-like disparities. As of late, Behera and Panda considered scalar variational-type disparities for single-

valued mappings on Hausdorff topological vector spaces, which was summarized structure variational-like inequality.

Leave X alone a Hausdorff topological vector space, Y a topological vector space, K a nonempty curved subset of X, and T : K → $2^{L(X'Y)}$,Ɵ : K × K → X, η] : K x K → Y be three mappings, where L(X, Y) is the space of all direct constant operators from X into Y. In, the creators got a summed up aftereffect of Minty's lemma and by utilizing it they considered the presence of answers for vector variational-type disparities for single-valued planning T: K ~ L(X,Y).

In , the creators got a summed up consequence of Minty's lemma and by utilizing it they considered the presence of answers for vector variational-type disparities for single-valued planning T: K → L(X,Y).

Let $\{C(x) : x \in K\}$ be a group of closed convex cones in Y. characterize a partial request $\leq_{C(x)}$ in Y with the convex cone C(x) as

For $y_1,\ y_2 \in Y,\ y_1 \leq_{C(x)} y_2,$ if and just if Y2 - Yl€C(x).

DEFINITION (1, 1),A mapping $f : K \to Y$ is convex if for each $x_1, x_2 \in K$ and $t \in (0,1),$

$$f(tx_1 + (1-t)x_2) \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

That is

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x).$$

We consider the accompanying Generalized Vector Variational-Type Inequality Problem (GVVTIP). Summed VECTOR VARIATIONAL-TYPE INEQUALITY PROBLEM. Find xo €K such that for all $y \in K$, there happens so $\in T(x_0)$ with the end goal that

$$\langle s_0, \theta(y, x_0)\rangle + \eta(x_0, y) \notin -\text{int}\, C(x_0),$$

where (s_0, υ) signifies the assessment of so at υ.

As end product, for T : K → L(X, Y), we consider the accompanying Vector Variational-Type Inequality Problem (VVTIP).

VECTOR VARIATIONAL-TYPE INEQUALITY PROBLEM. Find $x_0 \in K$ with the end goal that for all $y \in K$

$$\langle T(x_0), \theta(y, x_0)\rangle + \ \eta(x_0, y) \notin -\text{int}\, C(x_0).$$

Presently, we present the accompanying popular Fan's mathematical lemma.

LEMMA 1.1.) Let K be a nonempty conservative curved subset a Hausdorff topological vector space X. Let A be a subset of K x K fulfilling the accompanying conditions:

(1) for each $x \in K,\ (x,x) \in A,$,

(2) for each fixed $x \in K,$, the set $A_x = \{y \in K : (x,y) \in A\}$ is dosed in K,

(3) for each fixed $y \in K,$, the set A y = $\{x \in K : (x,y) \notin A\}$ is raised in K. there occurs an $x_0 \in K$ with the end goal that $K \times \{x_0\} \subset A.$.

DEFINITION 1.2. . Let X, Y be topological vector spaces and T : X → 2 ^{Y} a set-valued mapping.

(1) T is supposed to be Upper Semicontinuous (briefly, u.s.c.) at $x_0 \in X$ if for any open neighborhood N containing T(x_0) there exists a local M of $x_0$with the end goal that T(M) ⊂ N. T is said to be u.s.c, if T is u.s.c, at each point x € X.

(2) T is supposed to be closed at $x \in X$ if for each nets $\{x_\alpha\}$ meeting to x and $\{y_\alpha\}$ converging to y y with the end goal that $y_\alpha \in T(x_\alpha)$ for distress a, we have $y \in T(x).$ T is supposed to be dosed flit is is shut at each point $x \in X.$

DEFINITION 1.3. Let X, Y be topological vector spaces and T : X → 2 ^{Y} a set-valued mapping. T has a dosed diagram if the chart of $\{(x,y) \in X \times Y : y \in T(x)\}$ is shut in X × Y.

LEMMA 1.2. Let X, Y be topological vector spaces and T : X → 2 ^{y} be a set-valued mapping.

(1) If K is a reduced subset of X, and T is u.s.c, and conservative valued, then T(K) is smaller.

(2) If T is u.s.c, and reduced -valued, then T is shut.

2. RESULTS

Now we consider the presence hypothesis of answers for (GVVTIP).

THEOREM 2.1. Leave X alone a Hausdorff topological vector space, Y a topological vector space, K a nonernpty smaller arched subset of X, and {C(x) : x € K} a group of shut raised cones in Y. Let a set-valued planning W : K → 2 Y characterized by W(x) = Y \ {-int C(x)} have a shut chart, where int indicates the inside. Expect that T : K → 2 L(X'Y) is a u.s.c, planning with reduced qualities, and Θ : K × K → X and η : K × K → y are mappings with the end goal that.

there exists $s \in T(x)$ with the end goal that

$\langle s, \theta(x,x)\rangle + \eta(x,x) \notin -\operatorname{int} C(x),\ for\ all\ x \in K,$

(1) the operator

$x \longmapsto \langle s, \theta(x,y)\rangle + \eta(y,x)$

of K into Y is raised for all $y \in K$ and for all $s \in T(y),$

(2) the mappings

$y \mapsto \theta(\cdot, y) \quad \text{and} \quad y \mapsto \eta(y, \cdot)$

arenonstop,At that point (GVVTIP) is resolvable.

PROOF. Let A := {(x,y) E g × g : there exists s E T(y) with the end goal that $\langle s, \theta(x,y)\rangle + \eta(y,x) \notin$ -int C(y)}, then A is nonempty by (1).

For each fixed $x \in K,$

$A_x := \{y \in K : (x,y) \in A\}$

= {y € K: there exists s € T(y) with the end goal that $\langle s, \theta(x,y)\rangle + \eta(y,x) \notin -\operatorname{int} C(y)\}$

islocked. In fact, let $\{y_\lambda\}$ be a net in Ax with the end goal that $y_\lambda \to y_0$. Since $y_\lambda \in A_x,$, we have

Therehappens $s_\lambda \in T(y_\lambda),$ with the end goal that $\langle s_\lambda, \theta(x,y_\lambda)\rangle + \eta(y_\lambda, x) \in W(y_\lambda)$.

By Lemma 1.2(1), T(K) is reduced, and henceforth, without loss of over-simplification, we can accept that there exists $s_0 \in T(K)$ such that $s_\lambda \to s_0$. By Lemma 1.2(2), T is shut, and consequently,, $s_0 \in T(y_0)$.By condition (3) and W has a closed chart, we have

there exists $s_0 \in T(y_0),$ with the end goal that $\langle s_0, \theta(x,y_0)\rangle + \eta(y_0, x) \in W(y_0).$

Subsequently, yo€ A_x and A_x is shut.

Then again, for each fixed y € K,

$$A^y := \{x \in K : (x,y) \notin A\}$$

$$= \{x \in K : \text{for all } s \in T(y), \langle s, \theta(x,y)\rangle + \eta(y,x) \in -\text{int}\, C(y)\}$$

is convex. In point, let $x_1, x_2 \in A^y$ and t € (0,1), by situation (2), we have for all y € K and s € T(y),

$$[\langle s, \theta(tx_1 + (1-t)x_2, y)\rangle + \eta(y, tx_1 + (1-t)x_2)]$$

$$\leq_{C(y)} t\,[\langle s, \theta(x_1, y)\rangle + \eta(y, x_1)] + (1-t)\,[\langle s, \theta(x_2, y)\rangle + \eta(y, x_2)].$$

Subsequently,

$$t\,[\langle s, \theta(x_1, y)\rangle + \eta(y, x_1)] + (1-t)\,[\langle s, \theta(x_2, y)\rangle + \eta(y, x_2)]$$

$$-[\langle s, \theta(tx_1 + (1-t)x_2, y)\rangle + \eta(y, tx_1 + (1-t)x_2)] \in C(y).$$

Meanwhile

$$[\langle s, \theta(x_1, y)\rangle + \eta(y, x_1)] \in -\text{int}\, C(y)$$

Then

$$[\langle s, \theta(x_2, y)\rangle + \eta(y, x_2)] \in -\text{int}\, C(y),$$

We have

$$\langle s, \theta(tx_1 + (1-t)x_2, y)\rangle + \eta(y, tx_1 + (1-t)x_2) \in -\text{int}\, C(y).$$

What's more, thust$_{x1}$+ (1 - t)x$_2$€ $A^υ$, $A^υ$is curved. By Lemma 1.2, there exists a x_o € K with the end goal that K × {x_o} ⊂ A. This suggests there exists a x_o € K with the end goal that for all y € K, there exists s_o € T(xo) with the end goal that $\langle s_0, \theta(y, x_0)\rangle + \eta(x_0, y) \notin -\text{int}\, C(x_0).$

COROLLARY 2.2. ConsideringT : K→ L(X,Y) instead ofT : K → 2 $L^{(X,Y)}$ in Theorem 2.1, we can show the existence of solution to (VVITP).

COROLLARY 2.3. Considering a zero mapping ~ in Corollary 2.2, we obtain Theorem 2.2 in as a corollary.

In Theorem 2.1, we considered K to be a nonempty compact convex subset of a Hausdorff topological vector space X. But in the following theorem, we do not assume that K is compact.

THEOREM 2.4Leave X alone a Hausdorff topological vector space, Y a topological vector space, K a nonempty arched subset of X, and {C(x) : x € K} be a group of dosed curved cones in Y. Let a set-valued planning W : K → 2^y characterized by W(x) = Y \ {-int C(x)} have a shut diagram. H : K x K → Y are mappings fulfilling conditions (1)- (3) in Theorem 2.1, and the accompanying. (4) There exists a nonempty minimized curved subset D of K and with the end goal that for all x € K \ D, there exists s € T(x) with the end goal that is,

$$\langle s, \theta(u, x)\rangle + \eta(x, u) \in -\operatorname{int} C(x).$$

Then (GVVTIP) is resolvable in D.

PROOF. For all $x \in K$,

$B_x := \{y \in D$:there exists s € T(y) with the end goal that $\langle s, \theta(x, y)\rangle + \eta(y, x) \notin -\operatorname{int} C(y)\}$

is nonempty by (1). And for every x € K, let

C_x := {y € K: there exists s € T(y) with the end goal that $\langle s, \theta(x, y)\rangle + \eta(y, x) \notin -\operatorname{int} C(y)\}$,

we can show that C_x is shut by a similar strategy in the evidence of Theorem 2.1. Since D is shut in X, B_x - D ∩ C_x is shut subset of D. Obviously (GVVTIP) has an answer in D if $\cap_{x \in K} B_x \neq \phi$. For this, it is adequate to demonstrate that the family (B_x : x € K} has the finite crossing point property. Let $x_l, x_2, \ldots, x_n$ be discretionary finite components of K and let D_h = $(D \cup \{x_1, x_2, \ldots, x_n\})$where co indicates raised structure. At that point Dh is a compcat curved subset of K. By Theorem 2.1, there exists a x_0 € D_h with the end goal that for all y € D_h, there exists $s_0 \in T(x_0)$,

$$\langle s_0, \theta(y, x_0)\rangle + \eta(x_0, y) \notin -\operatorname{int} C(x_0). \quad \textbf{(7)}$$

It tends to be demonstrated that $x_0 \in D$.Actually, if $x_0 \notin D$, **at that point by condition (4), there exists** $u \in D$ **with the end goal that for such** $x_0 \in K \setminus D$, there exists $s_0 \in T(x_0)$, **with the end goal that** $\langle s_0, \theta(u, x_0)\rangle + \eta(x_0, u) \in -\operatorname{int} C(x_0)$, which negates (2.1), when $u = y$.Specifically, $x_0 \in C_{x_i}$, for all x_i . Indeed, if $x_0 \notin C_x$for some x_i, at that point for all $s \in T(x_0)$,

$$\langle s, \theta(x_i, x_0)\rangle + \eta(x_0, x_i) \in -\text{int}\, C(x_0). \quad \textbf{(8)}$$

In any case, since xi € D_h , from Theorem 2.1, we can pick $t \in T(x_0)$ with the end goal that

$$\langle t, \theta(x_i, x_0)\rangle + \eta(x_0, x_i) \notin -\text{int}\, C(x_0),$$

which repudiates (8). Henceforth, x_0 € B_{xi} for I = 1, 2,..., n. Accordingly,

$$\bigcap_{i=1}^{n} B_{x_i} \neq \emptyset.$$

Consequently, the family $\{B_x : x \in K\}$ has the finite crossing point property, so there exists y E D with the end goal that for each $x \in K$, there exists $s \in T(y)$ to such an extent that

$$\langle s, \theta(x, y)\rangle + \eta(y, x) \notin -\text{int}\, C(y).$$

Thusly, there exists a $x_0 \in D$ end goal that for all $y \in K$, there exists $s_0 \in T(x_0)$ such an extent that

$$\langle s_0, \theta(y, x_0)\rangle + \eta(x_0, y) \notin -\text{int}\, C(x_0).$$

Conclusion 2.5. Thinking about T : K → L(X,Y) rather than T : K →2 $^{L(X'Y)}$ in Theorem 2.4, we can show the presence of answer for (VVTIP).

End product 2.6. Thinking about a zero planning η in Corollary 2.5, we acquire Theorem 2.3 in as a culmination.

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