A REVIEW OF LEBESGUE MEASURABLE FUNCTION, LEBESGUE MEASURE AND INTEGRATION

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ABSTRACT :

If one recalls about area and volume, One definitely recalls from the simple formulas for the areas of rectangles and triangles to the quite sophisticated calculations with double and triple integrals in calculus. What One have probably never gone through, is a systematic theory for area and volume that unifies all the different methods and techniques.

In this paper we first study a unified theory for *d*-dimensional volume based on the notion of a *measure*, and then we shall use this theory to build a stronger and more flexible theory for integration. Instead of basing the calculation of volumes on integration, we shall create a theory of integration based on a more fundamental notion of volume.

And in the said process, this paper do review, Lebesgue's Measurable Function, Lebesgue's Measure and its integration.

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INTRODUCTION :

The theory will cover volume in R^d for all $d \in N$, including $d = 1$ and $d = 2$. To get a unified terminology, we shall think of the length of a set in R and the area of a set in R^2 as one- and two-dimensional volume, respectively.

To get a feeling for what we are aiming for, let us assume that we want to measure the volume of subsets $A \subset R^3$, and that we denote the volume of A by $\mu(A)$. What properties would we expext μ to have?

(i) $\mu(A)$ should be a nonnegative number or ∞. There are subsets of R^3 that have an infinite volume in an intuitive sense, and we capture this intuition by the symbol ∞.

(ii) $\mu(\emptyset) = 0$. It will be convenient to assign a volume to the empty set, and the only reasonable alternative is 0.

(iii) If $A_1, A_2, \ldots, A_n, \ldots$ are disjoint (i.e. non-overlapping)sets, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$. This means that the volume of the whole is equal to the sum of the volumes of the parts.

(iv) If $A = (a_1, a_2)\times(b_1, b_2)\times(c_1, c_2)$ is a rectangular box, then $\mu(A)$ is equal to the volume of A in the traditional sence, i.e.

$$\mu(A) = (a_2 - a_1)(b_2 - b_1)(c_2 - c_1)$$

It turns out that it is impossible to measure the size of *all* subsets of A such that all these requirements are satisfied; there are sets that are simply too irregular to be measured in a good way. For this reason we shall restrict ourselves to a class of *measurable sets* which behave the way we want. The hardest part of the theory will be to decide which sets are measurable.

We shall use a two step procedure to construct our measure μ: First we shall construct an *outer measure* μ^* which∈will assign a size $\mu^*(A)$ to *all* subsets $A \in R^3$, but which will not satisfy all the conditions (i)-(iv) above. Then we shall use μ^* to single out the class of measurable sets, and prove that if we restrict μ^* to this class, our four conditions are satisfied.

OUTER MEASURE IN R^d :

The first step in our construction is to define outer measure in R^d. The outer measure is built from rectangular boxes, and we begin by intoducing the appropriate notation and teminology.

Definition *A subset A of* R^d *is called an* open box *if there are numbers such that* $a_1^{(1)} < a_2^{(1)}, a_1^{(2)} < a_2^{(2)}, \ldots\ldots\ldots, a_1^{(d)} < a_2^{(d)}$

$$A = \left(a_1^{(1)}, a_2^{(1)}\right) \times \left(a_1^{(2)}, a_2^{(2)}\right) \times \ldots\ldots\ldots \times \left(a_1^{(d)}, a_2^{(d)}\right)$$

In addition, we count the empty set as a rectangular box. We define the volume $|A|$ *of* A *to be 0 if* A *is the empty set, and otherwise*

$$|A| = \left(a_2^{(1)} - a_1^{(1)}\right)\left(a_2^{(2)} - a_1^{(2)}\right) \ldots\ldots\ldots \left(a_2^{(d)} - a_1^{(d)}\right)$$

Observe that when $d = 1$, 2 and 3, A denotes the length, area and volume of A in the usual sense.

If A = $\{A_1, A_2, \ldots, A_n, \ldots\}$ is a countable collection of open boxes, we define its *size* |A| by

$$|A| = \sum_{k=1}^{\infty} |A_k|$$

(we may clearly have |A|=∞).Note that we can think of a finite collection A={A_1, A_2,……,A_n} of open boxes as a countable one by putting in the empty set in the missing positions: A = $\{A_1, A_2, \ldots, A_n, \emptyset, \ldots\}$. This is the main reason for including the empty set among the open boxes. Note also that since the boxes A_1,A_2,…..may overlap, the size |A| need not to be closely connected to the volume of $\bigcup_{n=1}^{\infty} A_n$.

A *covering* of a set $B \subset \mathrm{R}^d$ is a countable collection

$$A = \{A_1, A_2, \ldots, A_n, \ldots\}$$

of open boxes such that $B \subset \bigcup_{n=1}^{\infty} A_n$. We are now ready to define outer measure.

Definition *The* outer measure *of a set* $B \in \mathrm{R}^d$ *is defined by*

$$\mu^*(B) = inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by open boxes}\}$$

The idea behind outer measure should be clear – we measure the size of *B* by approximating it as economically as possible from the outside by unions of open boxes. You may wonder why we use open boxes and not closed boxes

$$A = \left[a_1^{(1)}, a_2^{(1)}\right] \times \left[a_1^{(2)}, a_2^{(2)}\right] \times \ldots \times \left[a_1^{(d)}, a_2^{(d)}\right]$$

in the definition above. The answer is that it does not really matter, but that open boxes are a little more convenient in some arguments. The following lemma tells us that closed boxes would have given us exactly the same result. You may want to skip the proof at the first reading.

Lemma *For all* $B \subset \mathrm{R}^d$,

$$\mu^*(B) = inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by closed boxes}\}$$

Proof: We must prove that

$$inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } open \text{ boxes}\} =$$

$$= inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } closed \text{ boxes}\}$$

Observe first that if $\mathcal{A}_0 = \{A_1, A_2 \ldots.\}$ is a covering of B by *open* boxes, we can get a covering $A = \{\bar{A}_1, \bar{A}_2 \ldots.\}$ of B by *closed* boxes just by closing each box. Since the two coverings have the same size, this means that

$$\mu^*(B) = inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } open \text{ boxes}\} \geq$$

$$\geq inf\{|\mathcal{A}| : \mathcal{A} \text{ is a covering of } B \text{ by } closed \text{ boxes}\}$$

To prove the opposite inequality, assume $\epsilon > 0$ is given . if $A=\{A_1,A_2,\ldots.\}$ is a covering of B by closed boxes, we can for each n find an open box $|\bar{A}_n| < |A_n| + \frac{\epsilon}{2^n}$. then $\bar{A} = \{\bar{A}_n\}$ is a covering B by open boxes, and $|\bar{A}| < |A| + \epsilon$. Since $\epsilon > 0$ is arbitrarily close in size, and hence

$$\inf\{|A| : A \text{ is a covering of } B \text{ by } open \text{ boxes}\} \leq$$

$$\leq \inf\{|A| : A \text{ is a covering of } B \text{ by } closed \text{ boxes}\}$$

Here are some properties of the outer measure:

Proposition *The outer measure μ^* on R^d satisfies:*

(i) $\mu^*(\emptyset) = 0$.

(ii) (Monotonicity) If $B \subset C$, then $\mu^*(B) \leq \mu^*(C)$.

(iii) (Subadditivity) If $\{B_n\}_{n\in \mathrm{N}}$ is a sequence of subsets of R^d, then

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \mu^*(B_n)$$

(iv) For all closed boxes

$$B = \left[b_1^{(1)}, b_2^{(1)}\right] \times \left[b_1^{(2)}, b_2^{(2)}\right] \times \ldots \times \left[b_1^{(d)}, b_2^{(d)}\right]$$

we have

$$\mu^*(B) = |B| = \left(b_2^{(1)} - b_1^{(1)}\right)\left(b_2^{(2)} - b_1^{(2)}\right) \dots \dots \dots .\left(b_2^{(d)} - b_1^{(d)}\right)$$

Proof: (i) Since $A = \{\emptyset, \emptyset, \emptyset, \dots\}$ is a covering of $\emptyset$, $\mu^*(\emptyset) = 0$.

(ii) Since any covering of C is a covering of B, we have $\mu^*(B) \le \mu^*(C)$.

(iii) If $\mu^*(B_n) = \infty$ for some $n \in N$, there is nothing to prove, and we may hence assume that $\mu^*(B_n) < \infty$ for all n. Let $s > 0$ be given. For each $n \in N$, we can find a covering $A_1^{(n)}, A_2^{(n)}, \dots .. B_n$ such that

$$\sum_{k=1}^{\infty} |A_k^{(n)}| < \mu^*(B_n) + \frac{\epsilon}{2^n}$$

The collection $\left\{A_k^{(n)}\right\}_{k,n\in N}$ of all sets covering is a countable covering of $\bigcup_{n=1}^{\infty} B_n$, and

$$\sum_{k,n\in N} \left|A_k^{(n)}\right| = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} |A_k^n|\right) \le \sum_{n=1}^{\infty}\left(\mu^*(B_n) + \frac{\epsilon}{2^n}\right)$$

$$= \sum_{n=1}^{\infty} \mu^*(B_n) + \epsilon$$

This mean

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) \le \sum_{n=1}^{\infty} \mu^*(B_n) + \epsilon$$

And since ϵ is an arbitrary, positive number, we must have

$\mu^*(\bigcup_{n=1}^{\infty} B_n) \le \sum_{n=1}^{\infty} \mu^*(B_n)$

(iv) since we can cover B by $B_\epsilon = \{B_\epsilon, ø, ø, \dots ..\}$ where

$B = \left(b_2^{(1)} + \epsilon, b_1^{(1)} - \epsilon\right) \times \left(b_2^{(2)} + \epsilon, b_1^{(2)} - \epsilon\right) \times \dots . \times \left(b_2^{(d)} + \epsilon, b_1^{(d)} - \epsilon\right)$

For any $\epsilon > 0$, we see that

$\mu^*(B) \le |B| = \left(b_2^{(1)} - b_1^{(1)}\right)\left(b_2^{(2)} - b_1^{(2)}\right) \dots \dots \dots \left(b_2^{(d)} - b_1^{(d)}\right)$

The opposite inequality,

$$\mu^*(B) \geq |B| = \left(b_2^{(1)} - b_1^{(1)}\right)\left(b_2^{(2)} - b_1^{(2)}\right) \dots \dots \dots \left(b_2^{(d)} - b_1^{(d)}\right)$$

may seem obvious, but is actually quite tricky to prove. We shall need a few lemmas to establish this and finish the proof.

Lemma *Assume that the intervals* (a_0, a_K), (b_0, b_N), (c_0, c_M) *are partioned*

$$a_0 < a_1 < a_2 < \dots < a_K$$

$$b_0 < b_1 < b_2 < \dots < b_N\, c_0 <$$

$$c_1 < c_2 < \dots < c_M$$

and let $\Delta a_k = a_{k+1} - a_k$, $\Delta b_n = a_{n+1} - n_n$, $\Delta c_m = c_{m+1} - c_m$. *Then*

$$(a_K - a_0)(b_N - b_0)(c_m - c_0) = \quad \sum_{k,n,m} \Delta a_k \Delta b_n \Delta c_m$$

where the sum is over all triples (k, n, m) *such that* $0 \leq k < K$, $0 \leq n < N$, $0 \leq m < M$. *In other words, if we partition the box*

$$A = (a_0, a_K) \times (b_0, b_N) \times (c_0, c_M)$$

into KNM smaller boxes $B_1, B_2, \dots, B_{KNM}$, *then*

$$|A| = \sum_{j=1}^{K\,N\,M} |B_j|$$

Proof: If you think geometrically, the lemma seems obvious — it just says that if you divide a big box into smaller boxes, the volume of the big box is equal to the sum of the volumes of the smaller boxes. An algebraic proof is not much harder and has the advantage of working also in higher dimensions: Note that since

$$a_K - a_0 = \sum_{k=0}^{K-1} \Delta a_k\,, b_N - b_0 = \sum_{n=0}^{N-1} \Delta b_n\,,\ c_M - c_0 = \sum_{m=0}^{M-1} \Delta c_m$$

We have

$$(a_K - a_0)(b_N - b_0)(c_M - c_0) = \left(\sum_{k=0}^{K-1} \Delta a_k\right)\left(\sum_{n=0}^{N-1} \Delta b_n\right)\left(\sum_{m=0}^{M-1} \Delta c_m\right) = \sum_{k,n,m} \Delta a_k \Delta b_n \Delta c_m$$

The next lemma reduces the problem from countable coverings to finite ones. It is the main reason why we choose to work with open coverings.

Lemma *Assume that* $A = \{A_1, A_2, \ldots, A_n, \ldots\}$ *is a countable covering of a compact set K by open boxes. Then K is covered by a finite number* $A_1, A_2, \ldots, A_n$ *of elements in* A.

Proof: Assume not, then we can for each $n \in$ N find an element $x_n \in K$ which does not belongs to $\bigcup_{k=1}^{n} A_k$. Since K is compact, there is a subsequence $\{x_{n_k}\}$ converging to an element $x \in K$. Since A is a covering of K, x must belong to an A_i. Since A_i is open, $x_{n_k} \in A_i$ for all sufficently large k. But this is impossible since $x_{n_k} \notin A_i$ when $n_k \geq i$.

We are now ready to prove the inequalities in the above proposition

Lemma *For all closed boxes*

$$B = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$$

we have

$$\mu^*(B) \geq |B| = (a_2 - a_1)(b_2 - b_1)(c_2 - c_1)$$

Proof: By the lemma above, it suffices to show that if $A_1, A_2, \ldots, A_n$ is a finite covering of B, then

$$|B| \leq |A_1| + |A_2| + \cdots \ldots . . +|A_n|$$

Let,

$$A_i = \left(x_1^{(i)}, x_2^{(i)}\right) \times \left(y_1^{(i)}, y_2^{(i)}\right) \times \left(z_1^{(i)}, z_2^{(i)}\right)$$

We collect all the x-coordinates $x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, \ldots \ldots x_1^{(n)}, x_2^{(n)}$ and rearrange them according to size :

$$x_0 < x_1 < x_2 < \ldots < x_I$$

Doing the same with the y- and the z-coodinates, we get partitions

$$y_0 < y_1 < y_2 < \ldots < y_J \; z_0 < z_1 < z_2 < \ldots < z_K$$

Let $B_1, B_2, \ldots, B_P$ be all boxes of the form $(x_i, x_{i+1}) \times (y_j, y_{j+1}) \times (z_k, z_{k+1})$ that is contained in at least one of the sets $A_1, A_2, \ldots, A_n$. Each A_i, $1 \leq i \leq n$ made up of a finite number of B_j's, and each B_j belongs to at least one of the A_i's.

$$|A_i| = |B_{ji_1}| + |B_{ji_2}| + \ldots \ldots . . +|B_{ji_q}|$$

Where B_{ji_1}, B_{ji_2}, …… , B_{ji_q} are the small boxes making up A_i. If we sum over all i, we get the following

$$\sum_{i=1}^{n} |A_i| > \sum_{j=1}^{P} |B_j|$$

(we get an inequality since some of the B_j's belong to more than one A_i, and hence are counted twice or more on the left hand side).

On the other hand, the B_j's almost form a partition of the original box B, the only problem being that some of the B_j's stick partly outside B. If we shrink these B_j's so that they just fit inside B, we get a partition of B into even smaller boxes $C_1, C_2, \ldots, C_Q$ (some boxes may disappear when we shrink them). Using Lemma 4.1.5 again, we see that

$$|B| = \sum_{k=1}^{Q} |C_k| < \sum_{j=1}^{P} |B_j|$$

So now we have,

$$|B| < \sum_{j=1}^{P} |B_j| < \sum_{i=1}^{n} |A_i|$$

And the Leema is proved.

We have now finally established all parts of Proposition 4.1.4. and are ready to move on. The problem with the outer measure $\mu*$ is that it fails to be countably additive: If A_n is a disjoint sequence of sets, we can only guarantee that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n)$$

And not that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right)_{n=1} = \sum_{n=1}^{\infty} \mu^*(A_n)_{n=1}$$

As it is impossible to change $\mu*$ such that the above equation holds for all disjoint se-quences $\{A_n\}$ of subsets of R^d, we shall follow a different strategy: We shall show that there is a large class M of subsets of R^d such that the above equation holds for all disjoint sequences where $A_n \in M$ for all n ϵ N. The sets in M will be called a "MEASURABLE SETS".

MEASURABLE SETS :

We shall now begin our study of measurable sets — the sets that can be assigned a "volume" in a coherent way. The definition is rather mysterious:

Definition 4.2.1 *A subset E of* $\mathbb{R}^d$ *is called* measurable *if*

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A)$$

for all $A \subset \mathbb{R}^d$. *The collection of all measurable sets is denoted by* M.

Although the definition above is easy to grasp, it is not easy to see why it captures the essence of the sets that are possible to measure. The best I can say is that the reason why some sets are impossible to measure, is that they have very irregular boundaries. The definition above says that a set is measurable if we can use it to split any other set in two without introducing any further irregularities, i.e. all parts of its boundary must be reasonably regular. Admittedly, this explanation is vague and not very helpful in understanding why the definition captures exactly the right notion of measurability. The best argument may simply be to show that the definition works, so let us get started.

Let us first of all make a very simple observation. Since $A = (A \cap E) \cup (A \cap E^c)$ subadditivity tells us that we always have

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \geq \mu^*(A)$$

Hence to prove that a set is measurable, we only need to prove that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$$

Our first observation on measurable sets is simple.

Lemma: *If E has outer measure 0, then E is measurable. In particular,* $\emptyset \in$ M.

Bevis: If E has measure 0, so has $A \cap E$ since $A \cap E \subset E$. Hence

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap E^c) \leq \mu^*(A)$$

for all $A \subset \mathbb{R}^d$.

Next we have:

Proposition 4.2.3 M *is an algebra of sets, i.e.:*

(i) $\emptyset \in$ M.

(ii) If $E \in$ M*, then* $E^c \in$ M.

(iii) If $E_1, E_2, \ldots, E_n \in$ M*, then* $E_1 \cup E_2 \cup \ldots \cup E_n \in$ M.

(iv) If $E_1, E_2, \ldots, E_n \in$ M*, then* $E_1 \cap E_2 \cap \ldots \cap E_n \in$ M.

Proof: We have already proved (i), and (ii) is obvious from the definition of measurable sets. Since $E_1 \cup E_2 \cup \ldots . \cup E_n = (E_1^c \cap E_2^c \ldots \ldots \cap E_n^c)^c$ by De Morgans laws, (iii) follows from (ii) and (iv). Hence it remains to prove (iv).

To prove (iv) is suffices to prove that if, $E_1, E_2 \in$ M, then $E_1 \cap E_2 \in M$ as we can then add more sets by induction. If we first use the measurability of E_1, we see that for any set $A \subset \mathbb{R}^d$

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)$$

Using the measurability of E_2, we get

$$\mu^*(A \cap E_1) = \mu^*((A \cap E_1) \cap E_2) + \mu^*((A \cap E_1) \cap E_2^c)$$

Combining these two expressions, we have

$$\mu^*(A) = \mu^*((A \cap (E_1 \cap E_2)) + \mu^*((A \cap E_1) \cap E_2^c) + \mu^*(A \cap E_1^c)$$

Observe that (draw a picture!)

$$(A \cap E_1 \cap E_2^c) \cup (A \cap E_1^c) = A \cap (E_1 \cap E_2)^c$$

and hence

$$\mu^*(A \cap E_1 \cap E_2^c) + \mu^*(A + E_1^c) \geq \mu^*(A \cap (E_1 \cap E_2)^c)$$

Putting this into the expression for $\mu^*(A)$ above, we get

$$\mu^*(A) \geq \mu^*((A \cap (E_1 \cap E_2)) + \mu^*(A \cap (E_1 \cap E_2)^c)$$

which means that $E_1 \cap E_2 \in$ M.

We would like to extend parts (iii) and (iv) in the proposition above to countable unions and intersection. For this we need the following lemma:

Lemma *If $E_1, E_2, \ldots, E_n$ is a disjoint collection of measurable sets, then*

$$\mu^*(A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)) = \mu^*(A \cap E_1) + \mu^*(A \cap E_2) + \ldots + \mu^*(A \cap E_n)$$

Proof: It suffices to prove the lemma for two sets E_1 and E_2 as we can then extend it by induction. Using the measurability of E_1, we see that

$$\mu^*(A \cap (E_1 \cup E_2)) = \mu^*((A \cap (E_1 \cup E_2)) \cap E_1) + \mu^*(A \cap (E_1 \cup E_2)) \cap E_1^c)$$

$$= \mu^*(A \cap E_1) + \mu^*(A \cap E_2)$$

Let us sum up our results so far.

Theorem 1: *The measurable sets M form a σ-algebra, i.e.:*

(i) $\emptyset \in$ M

(ii) If $E \in$ M, *then* $E^c \in$ M.

(iii) If $E_n \in$ **M** *for all* $n \in$ N, *then* $\bigcup_{n=1}^{\infty} E_n \in M$

(iv) If $E_n \in$ **M** *for all* $n \in$ N, *then* $\bigcap_{n=1}^{\infty} E_n \in M$

Proof : we have proved all except (iv) which fallows from (ii) and (iii) since

$\bigcap_{n=1}^{\infty} E_n = (\bigcup_{n=1}^{\infty} E_n^c)^c$.

Remark: By definition, a *σ*-algebra is a collection of subsets satisfying (i)- (iii), but — as we have seen — point (iv) follows from the others. There is one more thing we have to check: that *M* contains sufficiently many sets. So far we only know that M contains the sets of outer measure 0 and their complements! In the first proof it is convenient to use *closed* coverings as in Lemma to determine the outer measure.

Lemma *For each i and each* $a \in \mathrm{R}$*, the open halfspaces*

$H = \{(x_1, \dots, x_i, \dots, x_d) \in \mathrm{R}^d : x_i < a\}$ and $K = \{(x_1, \dots, x_i, \dots, x_d) \in \mathrm{R}^d : x_i > a\}$

Are measurable.

Proof: We only prove the H-part. We have to check that for any $B \subset \mathrm{R}^d$,

$$\mu^*(B) \geq \mu^*(B \cap H) + \mu^*(B \cap H^c)$$

Given a covering $\mathrm{A} = \{A_i\}$ of B by closed boxes, we can create closed- coverings $\mathrm{A}^{(1)} = \{A^{(1)}\}$ and $\mathrm{A}^{(1)} = \{A^{(2)}\}$ of $B \cap H$ and $B \cap H^c$, respectively, by putting

$$A^{(1)}_i = \{(x_1, \dots, x_i, \dots, x_d) \in A_i \ : \ x_i \leq a\}$$

$$A^{(2)} = \{(x_1, \dots, x_i, \dots, x_d) \in A_i \ : \ x_i \geq a\}$$

Hence,

$$|\mathrm{A}| = |\mathrm{A}^{(1)}| + |\mathrm{A}^{(2)}| \geq \mu^*(B \cap H) + \mu^*(B \cap H^c)$$

and since this holds for all closed coverings A of B, we get

$$\mu^*(B) \geq \mu^*(B \cap H) + \mu^*(B \cap H^c)$$

Lemma *All open boxes are measurable.*

Proof: An open box is a finite intersection of open halfspaces.

The next result tells us that there are many measurable sets:

Theorem 2: *All open sets in* R^d *are countable unions of open boxes. Hence all open and closed sets are measurable.*

Proof: Note first that the result for closed sets follows from the result for open sets since a closed set is the complement of an open set. To prove the theorem for open sets, let us first agree to call an open box

$$A = \left(a_1^{(1)}, a_2^{(1)}\right) \times \left(a_1^{(2)}, a_2^{(2)}\right) \times \ldots \ldots . \times \left(a_1^{(d)}, a_2^{(d)}\right)$$

$rational$ if all the coordinates $a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots \ldots ., a_1^{(d)}, a_2^{(d)}$ are are only countably many rationals boxes, and hence we only need to prove that if G is an open set, then

$$G = \bigcup \{B : B \text{ is a rational box contained in } G\}$$

We hereby leave the rest on the reader A measurable space is a set S, together with a nonempty collection, S, of subsets of S, satisfying the following two conditions:

1. For any A, B in the collection S, the set A − B is also in S.
2. For any A_1, A_2; · · · ϵ S, ∪A_i ϵ S.

The elements of S are called measurable sets. These two conditions are summarized by saying that the measurable sets are closed under taking finite differences and countable unions.

Think of S as the arena in which all the action (integrals, etc) will take place; and of the measurable sets are those that are "candidates for having a size". In some examples, all the measurable sets will be assigned a "size"; in others, only the smaller measurable sets will be (with the remaining meam surable sets having, effectively "infinite size").

Several properties of measurable sets are immediate from the definition.

1. The empty set ø, is measurable. [Since S is nonempty, there exists some measurable set A. So, A − A =ø is measurable, by condition 1 above.]
2. For A and B any two measurable sets, A ∩ B, A ∪ B, and A − B are all measurable.

It follows immediately, by repeated application of these facts, that the measurable sets are closed under taking any finite numbers of intersections, unions, and differences.

3. For A_1; A_2; · · · measurable, their intersection, ∩A_i, is also measurable. [First note that we have the following set-theoretic identity: A_1 ∩ A_2 ∩ A_3∩ · · · = A_1 − {(A_1 − A_2) ∪ (A_1 − A_3) ∪ (A_1 − A_4) ∪ · · ·}. Now, on the right, apply condition 1 above to the set-differences, and condition 2 to the union.] Thus, measurable sets are closed under taking countable intersections and unions.

Here are some examples of measurable spaces.

1. Let S be any set, and let S consist only of the empty set ø. This is a (rather boring) measurable space.

2. Let S be any set, and let S consist of all subsets of S. This is a measurable space.

3. Let S be any set, and let S consist of all subsets of S that are countable (or finite). This is a measurable space.

4. Let S be any set, and fix any nonempty collection P of subsets of S. Let S be the collection of subsets of S that result from the following construction. First set S = P. Now expand S to include all sets that result by taking differences and countable unions of sets in S. Next, again expand S to include all sets that result by taking differences and countable unions of sets in (the already expanded) S. Continue in this way, and denote by S the collection that results. Then (S; S) is a measurable space. Thus, you can generate measurable spaces by starting with any set S, and any collection P of subsets of S (i.e., those that you really want to turn out, in the end, to be measurable). By expanding that original collection P, as described above, you can indeed achieve a measurable space in which the chosen sets are indeed measurable.

5. Let $(S; S_1)$ be any measurable space, and let $K \subset S$ (not necessarily measurable). Let K_1 denote the collection of all subsets of K that are S-measurable. Then (K, K1) is a measurable space. [The two properties for (K, K1) follow immediately from the corresponding properties of (S, S1).] Thus, each subset of a measurable space gives rise to a new measurable space (called a subspace of the original measurable space).

6. Let (S^0, S^1) and $(S^{00}; S^{01})$ be measurable spaces, based on disjoint unu derlying sets. Set $S = S^0 \cup S^{00}$, and let S consist of all sets $A \subset S$ such that $A \cap S^1 \in S^1$ and $A \cap S^{01}$ ε S^{01}. Then (S, S_1) is a measurable space.

INTEGRATION OF NON-NEGATIVE SIMPLE FUNCTIONS :

You may think of a simple function as a generalized step function. The difference is that step functions are constant on intervals (in R), rectangles (in R^2), or boxes (in higher dimensions), while simple functions need only be constant on much more complicated (but still measurable) sets. Recall the definition of indicator functions I_A where A ϵ Σ. A mapping

f : S → R is said to be simple if it takes the form

$$f = \sum_{i=1}^{n} c_i I A_i$$

where $c_1, c_2, \ldots, c_n \in R$ and $A_1, A_2, \ldots, A_n \in \Sigma$ with $\bigcup_{i=1}^{n} A_i = S \ and\ A_i \cap A_j = \emptyset \ when\ i \neq j$. In other words, a simple function is a finite linear combination of indicator functions of non-overlapping sets. It follows from above theorems that every simple function is measurable. It is straightforward to prove that sums and scalar multiples of simple functions are themselves simple, so the set of all simple functions form a vector space. Recall that a mapping f : S→ R is non-negative if f(x) ≥ 0 for all x ϵ S, which in short is written as f ≥ 0. f ≤ g when g −f ≥ 0. It is easy to see that a simple function is non-negative if and only if $c_i \geq 0$ (1 ≤ i ≤ n).

Theorem 3 : Let f : S→R be measurable and non-negative. Then there exists a sequence (S_n) of non-negative simple functions on S with $s_n \leq s_{n+1} \leq f \ for\ all\ n \in N$ so that S_n converges pointwise to f as n →∞. If f is bounde then convergence is uniform.

Proof. This problem needs to be broken in three steps:

Step 1- Construction of (S_n).

Divide the interval [0,n) into $n2^n$ subintervals $\{I_j, 1 \leq j \leq n2^n\}$, each of length $1/2^n$ by taking $I_j = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$. Let $E_j = f^1(I_j)$ and $F_n = f^1([n,\infty))$. Then $S = \bigcup_{j=1}^{n2^n} E_j \cup F_n$. for all x ϵ S

$$S_n(x) = \sum_{j=1}^{n2^n} \left(\frac{j-1}{2^n}\right) 1_{E_j}(x) + n1_{F_n}(x).$$

Step 2 – Properties of (S_n)

For x ϵ E_j, $S_n(x) = (j-1)/2^n$ and $(j-1)/2^n \leq f(x) < (1/2^n)$ and so $S_n(x) \leq f(x)$. For x ϵ F_n, $S_n(x)$ = n and f(x) ≥ n. So it concludes that $S_n(x) \leq f$ for all n ϵ N. To show that $S_n \leq S_{n+1}$ fix an arbitrary j and consider $I_j = [(j-1)/2^n, j/2^n)$. For convenienace write I_j as I such that $I = I_1 \cup$

I_2 where I_1= $[(2j-2)/2^{n+1}, (2j-1)/2^{n+1})$ and I_2 = $[(2j-1)/2^{n+1}, 2j/2^{n+1})$. Let E= $f^{-1}(I)$, $E_1=f^{-1}(I_1)$ and $E_2=f^{-1}(I_2)$. Then $S_n(x) = (j-1)/2^2$ for all x ϵ E and so on for X ϵ E_1 and x ϵ E_2. It follows that $S_n \leq S_{n+1}$ for all x ϵ E.

Step 3 – Convergence of (S_n)

For any x ϵ S, since f(x) ϵ R there exists n_0 ϵ N so that f(x) ≤ n_0. Then for each n > n_0, f(x) ϵ I_j for some $1 \leq j \leq n2^n$. from here on the basis of above theorems, the result follows, from which the uniformity of convergence is deduced.

Now define the integral of a nonnegative simple function.

Definition : Assume that

$$f(x) = \sum_{i=1}^{n} a_i IA_i(x)$$

Is a non-negative simple function of standard form then, integral f is defined by

$$\int f\,(d\mu) = \sum_{i=1}^{n} a_i\mu(A_i)$$

Recall that 0.∞ = 0 hence $a_i\mu(A_i) = 0$ when $a_i = 0$ and $\mu(A_i) = \infty$. The integral of simple function is

$$\int I_A\,(d\mu) = \mu(A)$$

To see that the definition is reasonable, assume that you are in R^2. Since $\mu(A_i)$ measures the area of the set A_i, the product $a_i\mu(A_i)$ measures in an intuitive way the volume of the solid with base A_i and height a_i. We need to know that the formula in the definition also holds when the simple function is not on standard form. The step are the following, simple lemma

Lemma : If

$$g(x) = \sum_{j=1}^{n} b_j IB_j(x)$$

Where B_j's are disjoint and $R^d = \bigcup_{j=1}^{m} B_j$ then,

$$\int g\,d\mu = \sum_{j=1}^{n} b_j\mu(B_j)$$

Proof : the values b_1, b_2, ……, b_m need not be distinct but is easily fixed. let $b_{i,1}, b_{i,2}, \ldots, b_{i,n_i}$ be b_j's which are equal to C_i's(the distinct values taken by g), and let

$C_i = B_{i,1} \cup B_{i,2} \cup \ ....\cup B_{i,n_i}\ then\ \mu(C_i) = \mu(B_{i,1}) + (B_{i,2}) + \ ....+(B_{i,n_i})$ hence $\sum_{j=1}^{n} b_j\mu(B_j) = \sum_{i=1}^{k} c_i\mu(C_i)$ and now we have $\int g\, d\mu = \sum_{i=1}^{k} c_i\mu(C_i)$ and the lemma is proved.

Proposition *Assume that f and g are two nonnegative simple func- tions, and let c be a nonnnegative, real number. Then*

(i) $\int cf\, d\mu = c\int f\, d\mu$

(ii) $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$

Proof : of (i) we are already aware of hence left on the reader, to prove (ii) $f(x) = \sum_{i=1}^{n} a_i IA_i(x)$ and $g(x) = \sum_{j=1}^{n} b_j IB_j(x)$ be the standard representation of f and g and define $C_{i,j} = A_i \cap B_j$. By above lemma we have $\int f\, d\mu = \sum_{i.j} a_i\mu(C_{i.j})$ and $\int g\, d\mu = \sum_{i.j} b_j\mu(C_{i.j})$ and also

$$\int (f+g)\, d\mu = \sum_{i.j} (a_i + b_j)\mu(C_{i,j})$$

since the value of $f + g$ on $C_{i,j}$ is $a_i + b_j$. We can easily prove that the formula in the above Definition holds for all positive representations of step functions

Lemma : Assume that B is measurable set, b is positive real number, and $\{f_n\}$ an increasing sequence of non-negative simple function such that

$$\lim_{n\to\infty} f_n(x) \geq b\ \ for\ all\ x \in b$$

Then

$$\lim_{n\to\infty} \int_B f_n d\mu \geq b\mu(B)$$

Since $f_n(x) \uparrow b$ for all x ϵ B, and that the sequence $\{A_n\}$ is increasing and we habe

$$B = \bigcup_{n=1}^{\infty} A_n$$

We thus have,

$$\int_B f_n d\mu \geq \int_{A_n} a\ d\mu = am$$

Whenever $n \geq N$. since this hold good for any number of a less than b and m less than μ(B) we prove the required result.

Let (P,P_1) be a measurable space. A measure on (P, P_1) consists of a nonempty subset, M, of P_1, together with a mapping $M \xrightarrow{\mu} R^+$ (where R^+ denotes the sets of non-negative reals) satisfying the following two conditions.

from applying the above

1. For any A ϵ M and any B ⊂ A, with B ϵ P_1, we have B ϵ M.

2. Let A_1, A_2; · · · ϵ M be disjoint, and set A = A_1∪ A_2∪ · · ·. Then: This union A is in M if and only if the sum μ(A_1) + μ(A_2) + · · · converges; and when these hold that sum is precisely μ(A).

A set A ϵ M is said to have measure; and μ(A) is called the measure of A. Think of the collection M as consisting of those measurable sets that actually are assigned a "size" (i.e., of those size-candidates (in P_1) that were successful); and of μ(A) as that size.

Then the first condition above says that all sufficiently small measurable sets are indeed assigned size. The second condition says that the only excuse a measurable set A has for not being assigned a size is that " there is already too much measure inside A", i.e., that A effectively has "infinite measure". The last part of condition 2 says that measure is additive under taking unions of disjoint sets (something we would have wanted and expected to be true).

Several properties of measures are immediate from the definition.

1. The empty set ø is in M, and μ(ø) = 0. [There exists some set A ϵ M. Set B = ø and apply condition 1, to conclude ø ϵ M. Now apply condition 2 to the sequence (having union A = ø). Since A ϵ M, we have μ(ø) + μ(ø) + · · ·-μ(ø), which implies μ(ø) - 0.]

2. For any A, B ϵ M, A∩B, A∪B, and A−B are all in M. Furthermore, if A and B are disjoint, then μ(A∪ B) = μ(A) + μ(B). [The first and third follow immediately from condition 1, since A∩B and A−B are both subsets of A. For the second, apply condition 2 to the sequence A – B, B, ø,…….. of disjoint sets, with union A∪ B. Additivity of the measures also follows from this, since when A and B are disjoint, A − B = A.]

3. For any A, B ϵ M, with B ⊂ A, then μ(B)≤ μ(A). [We have, by the previous item, μ(A) = μ(B) + μ(A − B).] Thus, "the bigger the set, the larger its measure".

4. For any A_1, A_2; · · · ϵ M,∩Ai ϵ M. [This is immediate from condition 1 above, since ∩A_i ϵ P_1 and ∩A_i ⊂ A_1 ϵ M.]

Thus, the sets that have measure (i.e., those that are in M) are closed under finite differences, intersections and unions; as well as under countable intersections. What about countable unions? Let A_1, A_2; · · · be a sequence of sets in M, not necessarily disjoint. First note that $\cup A_i$ = A can always be written as a union of a collection of disjoint sets in M, namely of A_1, $A_2 - A_1$, $A_3 - A_2 - A_1$; · · ·. If the sum of the measures of the sets in this last list converges, then, by condition 2 above, we are guaranteed that A ϵ M. And if the sum doesn't converge, then we are guaranteed that A is not in M. Note incidentally, that convergence of this sum is guaranteed by convergence of the sum $\mu(A_1) + \mu(A_2) + \mu(A_3) +$ · · · (but, without disjointness, this last sum may exceed μ(A)). In short, the sets that have measure are not in general closed under countable unions, but failure occurs only because of excessive measure.

Here are some examples of measures.

1. Let P be any set, let P, the collection of measurable sets, be all subsets of P, let M= P, and, for A ϵ M, let μ(A) = 0. This is a (boring) measure.

2. Let P be any set, P all countable (or finite) subsets of P, M the collection of all finite subsets of P, and, for A ϵ M, let μ(A) be the number of elements in the set A. This is is called counting measure on P. Note that the set P itself could be uncountable.

3. Let P be any set and P the collection of all subsets of P. Fix a nonnegative function $P \xrightarrow{f} R^+$ on P. Now let M consist of all sets A ϵ S such that $\Sigma_A f$ converges. Thus, M includes all the finite subsets of P; and possibly some countably infinite subsets (provided there isn't too much f on the subset); and possibly even some uncountable infinite subsets (provided f vanishes a lot on the subset). For A ϵ M, set μ(A) = Σ_Af. This is a measure. For f = 1, it reduces to counting measure.

4. Let (P, P_1,M,μ) be any measurable space/measure. Fix any K ϵ P (not necessarily in P). Denote by K the collection of all sets in P that are subsets of K; and by M_K the

collection of all sets in M that are subsets of K. For A ϵ M_K, set $\mu_K(A) = \mu(A)$. Then (K, K_1, M_K, μ_K) is again a measurable space/measure. [This is an easy check, using for each property, the corresponding property of (P, P_1, M, μ).] Thus, any subset of the underlying set P of a space with measure gives rise to another space with measure. This is called, of course, a measure subspace.

5. Let (P^1, P^2, M^0, μ^0) and $(P^{11}, P^{22}, M^1, \mu^1)$ be measurable spaces/measures, with P^1 and P^{11} disjoint. Set $P = P^1 \cup P^{11}$; let P consist of $A \subset P$ such that $A \cap P^1 \in P^2$ and $A \cap P^{11} \in P^{22}$. Let P (resp, M) consist of $A \subset P$ such that $A \cap P^1 \in P^2$ and $A \cap P^{11} \in P^1$ (resp, $\in M^0$ and $\in M^1$). Finally, for $A \in M$, set $\mu(A) = \mu^0(A \cap P^1) + \mu^1(A \cap P^{11})$. This is a measurable space/measure. Thus, we may take the "disjoint union" of two measurable spaces/measures.

6. Let (P, P_1) be a measurable space, and let (M, μ) and (M, μ^0) be two measures on this space. [Note that they have the same M.] Define $M \xrightarrow{\mu+\mu^0} R^+$ by: $(\mu + \mu^0)(A) = \mu(A) + \mu^0(A)$. This is a measure, too. And, similarly, for any number a > 0, the mapping $M \xrightarrow{a\mu} R^+$ with action $(a\mu)(A) = a\mu(A)$ is a measure. Thus, we can add measures, and multiply them by positive constants.

We now obtain two results to the effect that "if a sequence of sets apa proaches (in a suitable sense) another set, then their measures approach the measure of that other set". In short, the measure of a set is "a continuous function of the set".

Theorem 4: Fix a measure space (P, P_1, M, μ), let $A_1 \subset A_2 \subset \cdots$ with $A_i \in M$; and set $A = \cup A_i$. Then: $A \in M$ if and only if the sequence $\mu(A_i)$ of numbers converges (as $i \to \infty$); and when these hold that limit is precisely $\mu(A)$.

Proof. Since the A_i are nested, we have the following set-theoretic identities:

$$A = A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \cup \cdots;$$

$$A_i = A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \cup \cdots \cup (A_i - A_i - 1):$$

Note that the sets in the unions on the right are disjoint, and in M. Since the union on the right of Eqn. (2) is finite, we have

$$\mu(A_i) = \mu(A_1) + \mu(A_2 - A_1) + \mu(A_3 - A_2) + \cdots + \mu(A_i - A_i - 1):$$

Hence: The $\mu(A_i)$ converge if and only if the sum $\mu(A_1) + \mu(A_2 - A_1) + \mu(A_3 - A_2) + \cdots$ converges; which in turn holds if and only if $A \in M$ and the definition of a measureg; and that when these hold $\mu(A) = \lim \mu(A_i)$ as per above equations and the definition of a measureg.

Theorem 5: Fix a measure space $(P; P_1; M; \mu)$, let $A_1 \supset A_2 \supset \cdots$, with $A_i \in M$; and set $A = \cap A_i$. Then $A \in M$, and $\lim_{i\to\infty} \mu(A_i) = \mu(A)$.

Proof. The proof is similar to that above (but easier), using the fact that $A_1 = A \cup (A_1 - A_2) \cup (A_2 - A_3) \cup \cdots$, where the sets on the right are disjoint, and in M.

As a final result on measure spaces, we show that, under certain circumc stances, a $(\mathcal{M}; \mu)$ that is "not quite a measure" can be made into one by including within $\mathcal{M}$ certain additional sets. Let (P, P_1) be a measurable space. Let $\mathcal{M}$ be a nonempty subset of P, and let μ be a mapping , $\mathcal{M} \xrightarrow{\mu} R^+$ Let us suppose that this (M; μ) satisfies the following two conditions:

1. For any $A \in M$ and any $B \subset A$, with $B \in P$, we have $B \in M$.

2. Let $A1, A2; \cdots \in M$ be disjoint, and set $A = A_1 \cup A_2 \cup \cdots$, their union. Then, provided $A \in M$, the sum $\mu(A_1) + \mu(A_2) + \cdots$ converges, to $\mu(A)$.

Thus, this (M; μ) is practically a measure on (P,P). Condition 1 above is identical to condition 1 for a measure; and condition 2 is only somewhat weaker than condition 2 for a measure. All that has been left out, in condition 2, is that portion of condition 2 that states: Whenever $\Sigma\mu(A_i)$ converges, then $A \in M$. That is, this (M; μ) is very nearly a measure, lacking only the requirement that disjoint unions of elements of M, if not too obese measurem wise, are themselves in M.

The present result is that, under the circumstances of the paragraph above, we can recover from that (M; μ) a measure. The idea is to enlarge the original M to include the missing sets. Denote by M^ the collection of all subsets of P that are of the form

[A_i,where A_1; A_2; · · · is a sequence of disjoint sets in M for which $\Sigma\mu(A_i)$ converges; and let $\mu'(A) = \Sigma\mu(A_i)$. Note that every set A in M is automatically in M'; with $\mu'(A) = \mu(A)$. This The present theorem is: This (M'; μ') is a measure.

The first step of the proof is to show that the function μ' is well-defined. To this end, let $A = A_1 \cup A_2 \cup \cdots$ be in M' via condition ii) above. Let $B_1, B_2, \cdots$ be a second disjoint collection of elements of M, with the same union: $\cup B_j = A$. We must show that $\Sigma\mu(B_j) = \Sigma\mu(A_i)$, i.e., that μ'(A), defined via the B_j, is the same as μ'(A) defined via the A_i. To see this, set, for i; j = 1; 2; · · ·, $C_{ij} = A_i \cap B_j$. Then the C_{ij} are disjoint and in M, and their union is precisely A. But by condition 2 we have $\Sigma_i\mu(C_{ij}) = \mu(B_j)$ and $\Sigma_j\mu(C_{ij}) = \mu(A_i)$. That $\Sigma\mu(A_i) = \Sigma\mu(B_j)$ follows.

To complete the proof, we must show that (M'; μ') satisfies conditions 1 and 2 for a measure. For condition 1: Let $A \in M'$: We have $A = \cup A_i$, where the A_i are disjoint and are in M, and are such that $\Sigma\mu(A_i)$ converges. Let $B \subset A$, with $B \in S$. We must show that $B \in M'$. But this follows, since $B = \cup (B \cap A_i)$, where the $B \cap A_i$ are disjoint, are in M, and are such that $\Sigma\mu(B \cap A_i)$ converges. We leave condition 2 as an (easy) exercise. Here is an example of an application of this result. Let $P = Z^+$, the set of positive integers, let P consist of all subsets of P, let M consist of all finite subsets of P, and, for $A \in M$, let $\mu(A) = \Sigma_{n \in A}(1/2^n)$, where the sum on the right is finite. This (M; μ) satisfies conditions 1 and 2 above. But it is not a measure, for it does not satisfy condition 2 for a measure space. In this case, the M' constructed above consists of all subsets of P, and, for $A \in M'$, $\mu'(A) = \Sigma_{n \in A}(1/2^n)$, where now the sum on the right is over the (possibly infinite) set A. The measure space (M, μ) here constructed will be recognized as a special case of Example above.

Finally, we remark that, when the original (M, μ) of the previous page happens to be a measure, then M' = M, and μ' = μ.

We now turn to what is certainly the most important example of a measure space: Lebesque measure. Let P = R, the set of reals. [The case $P = R^n$ is virtually identical, line-for-line, to this case; but P = R makes writing easier.]

Set I = (a; b), an open interval in R. The idea is that we want this interval to be measurable, with measure its length: $\mu(I) = b - a$. Let's try to turn this idea into a measure space. By condition 2 for a measure space, our collection M will have to include also sets of the form $K = I_1 \cup I_2 \cup \cdots$, a union of disjoint intervals, with measure $\mu(K) = \mu(I_1) + \mu(I_2) + \cdots$ provided the sum on the right converges. And furthermore, by condition 1 for a measure space, M will also have to include differences of intervals, i.e., the half-closed intervals $[a, b)$ and $(a, b]$, with measures again $b - a$. So, we expand our original M to include these new sets. Next, let us return, with this new, expanded M, to condition 2. By this condition, M must include also countable unions of the half-closed intervals. Returning to condition 1, we find that our M must include differences of these unions. Continue in this way, at each stage expanding the then-current M by including the new sets demanded by conditions 2 and 1. Does this process terminate? That is, do we, eventually, reach a point at which applying conditions 2 and 1 to the then-current M does not result in any further expansion of M? If this did occur, then we would be done. Presumably, we would at that point be able to write down some general form for a set in this final M, as well as a general formula for its measure. We would thus have our measure space. But, unfortunately, it turns out that this process does not terminate: Each passage through condition 2 and condition 1 requires that additional, new sets be included in M. In short, this is not a very good way to construct our measure space. So, let's try a new strategy. Fix any set $X \subset R$. Let $I_1, I_2, \cdots$ be any countable collection of open intervals that covers X [i.e., that are such that $X \subset \cup I_i$. Note that we do not require that the I_i be disjoint.] There always exists at least one such collection, e.g., $(-1, 1), (-2, 2), \cdots$. Now set $m = \Sigma\mu(I_i)$, the sum of the lengths of the I_i. This m is either a nonnegative number or "∞" (in case the sum fails to converge). We define the outer measure of X, written $\mu^*(X)$ to be the greatest lower bound of these m's, taken over all countable collections of open intervals that cover X; so $\mu^*(X)$ is either a nonnegative number, or "∞" (in case X is covered by no countable collection of intervals the sum of whose lengths converges). The outer measure of X reflects \how much open-interval is required to cover X", i.e., is a rough measure of the "size" of X. For example, for X already an interval, $X = (a; b)$, we have $\mu^*(X) = (b - a)$, its length (an assertion that seems rather obvious, but is in fact a bit tricky to prove). As a second example, let X be the set of rational numbers. Order the rationals in any way, e.g., 3/5, −398/57, 3; · · ·. Now fix any $\varepsilon > 0$. Let I_1 be the interval of length ε centered on the first rational (3/5); I_2 the interval of length $\varepsilon/2$ centered on the second rational (−398/57); and so on. Then these I_i cover X; and $\mu(I_1) + \mu(I_2) + \cdots = \varepsilon + \varepsilon/2 + \cdots = 2\varepsilon$. But $\varepsilon > 0$ is arbitarary: Thus, there exists a covering of X (the rationals) by open intervals the sum of whose lengths is as close to zero as we wish. We conclude: $\mu^*(X) = 0$. The same holds for any countable (or finite) subset of the reals. The outer measure has the sort of behavior we might expect of a measure. For example: For $X \subset Y \subset R$, then $\mu^*(X) \leq \mu^*(Y)$ (which follows from the fact that any covering of Y is already a covering of X). For X; Y $\subset$ R, $\mu^*(X \cup Y) \leq \mu^*(X) + \mu^*(Y)$ (which follows from the fact that the intervals in a covering of X taken together with the intervals in a covering of Y yields a collection of intervals that covers $X \cup Y$). Thus, it is tempting to try to construct our measure space using outer measure: Let M consist of all subsets X of P = R with finite outer measure, and set $\mu(X) = \mu^*(X)$. But, unfortunately, this does not work, as the following example illustrates. For a and b and two numbers in the interval [0; 1), write a∼b provided a-b is a rational numer. This is an equivalence trlation. Now suppose, for contradiction, that we had a measure space based on outer measure. By the first two properties above, we would have $\Sigma\mu^*(X_r) = \mu^*([0; 1)) = 1$, where the sum on the left is over all rationals r ϵ [0; 1).

Thus, the outer measure is somewhat flawed as a representative of the"\size" of a set, in the following sense. Certain sets (such as the X above) are, roughly speaking, so frothy that they cannot be covered efficiently by open intervals, and for these the outer measure is “too large".

This observation is the key to finding our measure space. For X and Y any two subsets of P = R, set $d(X; Y) = \mu^*(X - Y) + \mu^*(Y - X)$, so d(X; Y) is a nonnegative number (or possibly “1"). Think of d(X, Y) as reflecting the extent to which X and Y differ as sets", i.e., as an effective “distance" between the sets X and Y.

This interpretation is supported by the following properties:

1. We have d(X, Y) = 0 whenever X = Y. [But note, that the converse fails, e.g., with Y consisting of X together with any one number not in X.]

2. For any subsets X,Y, Z of R, we have d(X, Z) $\leq$ d(X,Y) + d(Y,Z). This follows from the facts that X − Z ⊂ (X − Y) ∪ (Y − Z) and Z − X ⊂ (Z − Y) ∪ (Y − X). That is, d(‘) satisfies the triangle inequality.

3. For any subsets X, X^0, Y, Y^0 of R, d(X ∪ Y, X^0 ∪ Y^0) $\leq$ d(X, X^0) + d(Y, Y^0), and similarly with " ∪" replaced by " ∩ " or "-"{This follows from the fact that the set-difference of X∪ Y and X^0 ∪ Y^0 is a subset of (X − X^0) ∪ (Y − Y^0); and similarly for " ∩ " and "-"{That is, nearby sets have nearby unions, intersections, and differences", i.e., the set operations are " continuous" as measured by d(,).

4. For any subsets X, Y of R, |μ^* (X)− μ^* (Y)| $\leq$ d(X, Y). This follows from X∪ (Y − X) = Y and Y ∪ (X − Y) = X. That is, outer measure is a d(;)-continuous function of the set.

As we have remarked, the outer measure is sometimes \too large", and this fact renders it unsuitable as a measure. But the outer measure is suitable for generating an effective distance, d(,), between sets, for in this role its propensity to be

"too-large" becomes merely an excess of caution.

We now turn to the key definition. Denote by M the collection of all subsets A of P = R with the following property: Given any ϵ > 0, there exists a K ⊂ R, where K is a finite union of open intervals, such that d(A; K) $\leq$ ϵ. And, for A ϵ M, set μ(A) = μ^* (A). In other words, the elements of M are the sets that can be "approximated" (as measured by d(,)) by finite unions of open intervals. And, similarly, μ(A) is approximated by the sum of the lengths of the intervals in K (as follows from the fact that d(A, K) $\leq$ ϵ implies |μ^*(A) − μ^*(K)| $\leq$ ϵ). It follows, in particular, that μ(A) is not "∞".

In the land of measure spaces, the more sets that are measurable the better. Do there exists measures that are better, in this sense, than Lebesque measure? That is, does there exist a measure (M';μ') on R that is an extene sion of Lebesque measure, in the sense that M' is a proper superset of M, and μ' agrees with μ on M? It turns out that there does. Let X denote any non-measurable set of finite outer measure. Let P' consist of all subsets of R of the form (A∩ X) ∪ (B − X), where A and B are measurable. Thus, for example, choosing A = B we conclude that P' ⊃ P; and, choosing A ⊃ X and B

=ø, we conclude that X ϵ P'. This collection is closed under differences and countable unions (as follows immediately from the fact that P is). Let M' ⊂ P consist of those sets of this form with B having finite measure; and, for any such set, set μ((A\X) [(B − X)) = μ^*(A ∖X)+μ(B) −μ^*(B ∖X). Thus, for example, X ϵ M', with μ'(X) = μ^*(X); and, for A ϵ M, μ'(A) = μ(A). One checks that this (M' ; μ') is indeed a measure space, and that it is indeed an extension of Lebesque measure. Since X ϵ M' but X ϵ M, this is a proper extension.

INTEGRALS OF NON-NEGATIVE FUNCTIONS :

We are now ready to define the general non-negative measurable functions. Let (S, S1) be a measurable space (not necessarily Lebesque). A real-valued function, $S \xrightarrow{f} \boldsymbol{R},$

is said to be measurable provided for O any open set in the reals f^{-1}[O] is measurable. This definition is hauntingly similar to that of continuity, the essential difference being that, since the domain in this case is a measurable space instead of a topological space, we require measurability, instead of open-ness, of f^{-1}[O]. As an example of this notion, we note

that every c ontinuous function, $R \xrightarrow{f} R$ is a Lebesque-measureable (for f containuous, each f^{-1}[O] ⊂ R is open, and hence Lebesque-measurable). As a second example, let, in a general measurable space, A1, · · · ;As be a finite number of disjoint measurable sets, whose union is S. Fix numbers a1; · · · ;as. Let f be the function such that f(x) = ai whenever x ϵ Ai, for i = 1; · · ·; s. Thus, this function is constant on each of the elements of a finite, measurable partition of S. This function f is measurable. A measurable function f with finite range (i.e., a function of the form above) is called a step function.

Here are two elementary properties of measurable functions. For the first, let $S \xrightarrow{f} R$ of S, that inverse images, under f, of differences of open sets (in R); of countable unions of such differences; of differences of such countable unions; etc., are all measurable. For the second property, let $S \xrightarrow{f} R$ *and* $S \xrightarrow{g} R$ be measurable. Then U = { (α, β)|α + β ϵ O| is open in R^2, whence $(f + g)^{-1}$[O] = w^{-1}[U] is measurable. More generally, any continuous function applied to two (or, indeed, to any finite number) of measurable functions is measurable. We note also that any continuous function applied to two (or any finite number) of step functions is a step function.

We are now ready to do integrals. Let (S, S, M, μ) be a measure space. Let us, for the present discussion, restrict ourselves to the following case: The set S itself has finite measure. We shall relax this assumption (which is made solely to avoid, at this point of the discussion, possibly divergent integrals) shortly.

Let f be any step function (with values a1, · · · ;as on measurable (and, by the assumption above, finitely measurable) sets A1; · · · ;As). We define the integral of this function f (over S, with respect to measure μ) byRSf dμ = a1μ(A1) + · · · + asμ(As). We note that this is the right sort of expression to be the integral: It multiplies the value of f by the size of the region on which f takes that value, and sums over the (finite number of) different values that a step function can assume. As an example, let our measure space be Lebesque measure on (0; 1), and let f be the (step) function with value 1 on the rationals (in (0; 1)), 0 on the irrationals. The integral of this function is zero (since the Lebesque measure of this set of rationals is zero).

This integral has the properties we would expect of an integral. For f any step function, $\int_s (af)\, d\mu = a\int_s f\, d\mu\{ for\ \sum(aai)\mu(Ai) = a\sum(ai)\mu(Ai)$ furthermore fo f and g any two step functions we have $\int_s (f+g) d\mu = \int_s f\, d\mu + \int_s g\, d\mu$ {let f have values a1,…..,as on A1,….,As and g have values b1,…..,bt on B1,….,Bt. Then (f+g) have the values ai+bj on $Ai \cap Bj,$ for i=1,…,s and j = 1,…,t. so $\int_s (f+g) d\mu = \sum_{ij}(ai+bj)\mu(Ai \cap Bj) =$ $ijai\mu Ai \cap Bj + ijbj\mu Ai \cap Bj.$ Now carry out the j-sum in the first term on the right, and the i-sum in the second. In short, the integral is linear in the step-function integrated. A further, key, property of this integral is that " small (step) functions have small integral".

We now expand the applicability of our integral from step functions to all bounded, measurable functions on our finite measure space. This expansion is based on the following key fact.

Theorem 6: Let be $S \xrightarrow{f} R$ be a bounded measurable function on (S,S1) then for every $\epsilon > 0$

there exists a step function g on (S, S_1) such that $|f - g| \leq \epsilon$.

Proof: Let I1; · · · ;Is be a finite collection of disjoint half-open intervals whose union covers the range of f. For each i, set $Ai = f^{-1}[Ii] \in S$; and choose $ai \in Ii$. Then the function g on S with value ai in Ai is a step function, and satisfies $|f - g| \leq \epsilon$.

Now fix any bounded measurable function f on our finite measure space. Fix a sequence of ϵ's approaching zero, and, for each, choose a step function $g\epsilon$ with $|f - g\epsilon| \leq \epsilon$. Then (since $|g\epsilon - g\epsilon 0| \leq (\epsilon + \epsilon^0)$) we have $|\int_s g\epsilon d\mu - \int_s g\epsilon 0 d\mu \leq (\epsilon + \epsilon^0)\mu(S)$, whence the $\int_s g\epsilon d\mu$ form a Cauchy sequence. By the integral of f (over S, with respect to measure μ), we mean the number to which this sequence converges: $\int_s f\, d\mu = \lim_{\epsilon \to 0} \int_s g\epsilon\, d\mu$ (noting that this number is independent of the particular sequence of g_ϵ chosen).

let (S,S_1,M,μ) be a finite measure space, and let $A \subset S$ be measurable. Then, as we have seen earlier, we recover, in the obvious way, a measure space based on A: The measurable sets in this space are the measurable subsets of A; and the measure of such a set is simply its μ-measure. Next, let f be any measurable function on S. Then the restriction of f to A is a measurable function on this measure (sub-)space. Furthermore if the original function f on S was bounded, then its restriction to A is also bounded. Under these circumstances (f a bounded measurable function on a finite measure space and $A \subset S$ measurable) we define the integral of f over A, written $\int_A f\, d\mu$ to be the integral over the measure (sub-)space A of the function f-restricted- to-A.

A key property of this integral is that it gives rise to a countable additive set function, in the following sense. Let A 1 , A 2 . . . . be disjoint measurable sets, with $\cup Ai = S$. Then, we claim, $\sum \int_{Ai} f\, d\mu = \int_s f\, d\mu$ It suffices to prove this claim for the case in which f a step function (for replacing, in the formula above, a general bounded, measurable function f by a step function g with $|f - g| \leq \epsilon$ changes neither side of that equation by more than $\epsilon\mu(S)$). So, let f be the step function taking values b1, b2, · · · ,bs on disjoint measurable sets B1, B2, · · · ,Bs. Consider the expression $\sum bj\mu(Ai \cap Bj)$, where the sum is over i = 1,2; ·

· ·,s and j = 1, 2, · · ·, s. Fixing j and carrying out the sum over i, we obtain bj μ(Bj) while fixing i and carrying out the sum over j, The result follows.

To summarize, we first defined the integral of a step function over a finite measure space (by the obvious formula); and then defined the integral of an arbitrary bounded measurable function over that space (by approximating that function by step functions). We now wish to relax the conditions that the measure space be finite; and that the function be bounded. In doing so, we shall encounter a new phenomenon: Our integrals may in some cases (i.e., for some functions and some regions of integration) fail to converge.

Let (S,S1,M,μ) be an any measure space; and let f be a measurable function thereon. We assume for the present that this f is nonnegative (an assumption we shall relax in a moment). Next, denote by M^0 the collection of all measurable A ⊂ S having finite measure, and on which f is bounded. [Note that there are many such sets: Take, e.g., f^{-1}[(a; b)], and, if this set fails to have finite measure, take any finite-measure subset of it.] For any such A ϵ M^0, set $\mu^0(A) = \int_A f d\mu$, noting that the integral on the right makes sense. Now, this (M^0, μ^0) that we have just constructed will not in general be a measure space. While it always satisfies condition 1 of that definition, it may fail to satisfy condition 2. However, this (M^0,μ^0) does satisfy the conditions 1 and 2 listed above. [Condition 1 is immediate, while condition 2 is what we just showed two paragraphs ago.] As we showed, we may, under these conditions, expand (M^0, μ^0) to a measure space. That construction, in more detail, is the following. Consider any sequence, A1, A2, · · · of disjoint sets in M0 (so each Ai has finite measure, and on each f is bounded), such that $\sum \mu^0(Ai)(= \sum \int_A f\, d\mu$ converges.

Let $M^{\wedge 0}$ consist of all subsets of S given as the union of such Ai; and, for $A = \cup Ai \in M^{\wedge 0}$, set $\mu^{\wedge 0} = \Sigma \mu^0(Ai)$. Then, as we showed earlier, this ($M^{\wedge 0}$, $\mu^{\wedge 0}$) is a measure space. We say that the measurable function f ≥ 0 is integrable

over A ⊂ S provided A ϵ $M^{\wedge 0}$; and, for such a set A, we write $\int_A f\, d\mu = \mu^0(A)$ Thus, a general nonnegative measurable function on a general measure space is integrable over measurable A ⊂ S provided that A can be written as a countable union of disjoint sets, each of finite measure and on which f is bounded, such that the sum of the integrals of f over those sets converges; and in this case the integral is given by that sum.

It is easy, finally, to relax the condition $f \geq 0$ (which was made Solely in order to invoke the earlier result on constructing measure spaces).

Any measurable function f can be written uniquely as $f^+ - f^-$ where $f^+ \geq 0$ and $f^- \geq 0$ are measurable function. We say that f is integrable on $A \subset S$ provided both f^+ and f^- are.

This integral inherits from its predecessors linearity in the function.

So, the subject of integration (of arbitrary measurable functions over ara bitrary measure spaces) is a remarkably simple one. We progress in turn from integration of step functions over finite measure spaces (the obvious formula) to integration of bounded functions over finite measure spaces (as limits of step-function integrals) to integration of nonnegative measurable functions over arbitrary measure spaces (restricting the region of integration to achieve convergence of the integral) to integration of arbitrary measurable functions over arbitrary measure spaces (writing such a function as the difference of nonnegative functions). It is true that some work has to be invested at the beginning, to get the notion of a measure in the first place. But, once the ground is prepared, things go easily and smoothly. General integration thet ory is much simpler, much more general, and much more useful than the theory of the Riemann integral.

Definition : if $f : R^d \rightarrow [0, \infty]$ is measurable, we define

$\int f \, d\mu = sup\{\int g \, d\mu | g \text{ is a nonegative simple function}, g \leq f\}$

Remark: Note that if f is a simple function, we now have two definitions of $\int f \, d\mu$; the original one in above Definition and a new one in the definition above. It follows from above Proposition that the two definitions agree. The definition above is natural, but also quite abstract, and we shall work toward a reformulation that is often easier to handle.

Proposition : Let $f : R^d \rightarrow [0, \infty]$ be measurable and assume that $\{h_n\}$ is an increasing sequence of simple function covering pointwise to f, then

$$\lim_{n\to\infty} \int h_n d\mu = \int f \, d\mu$$

Proof : From above we already have $\int h_n d\mu \leq \int f \, d\mu$ for all n , hence we must also have

$$\lim_{n\to\infty}\int h_n d\mu \le \int f\, d\mu$$

To get the opposite inequality it suffice to show that

$$\lim_{n\to\infty}\int h_n d\mu \ge \int g\, d\mu$$

The proposition above would lose much of its power if there weren't any increasing sequences of simple functions converging to f . The next result tells us that there always are. The shown argument, is a key to why the theory works.

Proposition : If f : $R^d \to [0, \infty]$ is measurable then there $\{h_n\}$ is an increasing sequence of simple function covering pointwise to f. Moreover for each n either

$$f_n(x) - \frac{1}{2^n} < h_n(x) \le f_n(x) or\ \ h_n(x) = 2^n$$

Proof : To construct the simple function h_n we cut interval $[0, 2^n)$ into half open sub interval of length $1/2^n$ i.e.

$$I_k = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

Where $0 \le k \le 2^{2n}$ and let

$$A_k = f^{-1}(I_k)$$

We now define

$$\Box_n(x) = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} IA_k(x) + 2^n I_{\{x|f(x)\ge 2^n\}}$$

By definition, h_n is a simple function no greater than f. Since the intervals get narrower and narrower and cover more and more of [0, ∞), it is easy to see that h_n converges pointwise to f . To see why the sequence increases, note that each time we increase n by one, we split each of the former intervals I_k in two, and this will cause the new step function to equal the old one for some x's and jump one step upwards for others. The last statement follows directly from the construction.

Remark: You should compare the partitions in the proof above to the par- titions you have seen in earlier treatments of integration. When we integrate a function of one variable in calculus, we partition an interval $[a, b]$ on the *x*-axis and use this partition to approximate the original function by a step function. In the proof above, we instead partitioned the *y*-axis into intervals and used this partition to approximate the original function by a simple function. The difference is that the latter approach gives us much better control over what is going one; the partition controls the oscillations of the function. The price we have to pay, it that we get simple functions instead of step functions, and to use simple functions for integration, we need measure theory.

Proposition : Assume that f, g : $R^d \to [0, \infty]$ are measurable function and that c is a nonnegative real number, then

(i) $\int cf\, d\mu = c \int f\, d\mu$

(ii) $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$

(iii) If g ≤ f, then $\int g\, d\mu \leq \int f\, d\mu$

Proof : (iii) is immediate from the definition and (i) is left to the reader. To prove (ii), let $\{f_n\}$ and $\{g_n\}$ be increasing sequence of simple function converging to f and g, respectively. Then $\{f_n + g_n\}$ is an increasing sequence of simple function converging to $f+g$, and

$$\int (f+g) d\mu = \lim_{n-\infty} \int (f_n + g_n) d\mu = \lim_{n-\infty} \left(\int f_n d\mu + \int g_n d\mu \right) =$$

$$= \lim_{n-\infty} \int f_n d\mu + \lim_{n-\infty} \int g_n d\mu = \int f d\mu + \int g d\mu$$

One of the big advantages of Lebesue integration over traditional Riemamm integration , is that Lebesgue integral is much better behaved with respect to limits.

Monotone Convergence Theorem : If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions such that

$f(x) = \lim_{n\to\infty} f_n(x)$ for all x

Then,

$$\lim_{n\to\infty} \int f_n d\mu = \int f \, d\mu$$

i.e.,

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n \, d\mu$$

Proof : recalling from the above observations, since $f_n \leq f$, hence $\int f_n d\mu \leq \int f \, d\mu$ for all n, so we hereby get

$$\lim_{n\to\infty} \int f_n d\mu \leq \int f \, d\mu$$

To prove the opposite inequality, we approximate each f_n by simple functions, let $\{h_n\}$ be its n-th approximation. If the sequence $\{h_n\}$ converges to f , we have

$$\lim_{n\to\infty} \int h_n d\mu = \int f \, d\mu$$

Now, since $f_n \geq h_n$

$$\lim_{n\to\infty} \int f_n d\mu \geq \int f \, d\mu$$

To show that $h_n(x) \to f(x)$, we have from above that

$$f_n(x) - \frac{1}{2^n} < h_n(x) \leq f_n(x)$$

If $h_n(x) = 2^n$ for infinitely many n then $h_n(x)$ goes to ∞ and hence to f(x). if, $h_n \neq 2^n$ then eventually we have $f_n(x) - \frac{1}{2^n} < h_n(x) \leq f_n(x)$ and $h_n(x)$ converges to f(x) and consequently we have

$$\lim_{n\to\infty}\int f_n d\mu = \int \lim_{n\to\infty} f_n \, d\mu$$

Fatou's Lemma : Assume that $\{f_n\}$ is a sequence of nonnegative measurable function . Then

$$\lim_{n\to\infty}\inf\int f_n d\mu \geq \int \lim_{n\to\infty}\inf f_n \, d\mu$$

Proof : Let $g_k(x) = inf_{k\geq n} f_n(x)$. Then $\{g_k\}$ is an increasing sequence of nonnegative measurable function and by Monotone Convergence Theorem we finally have

$$\lim_{k\to\infty}\int g_k d\mu = \int \lim_{k\to\infty} g_k \, d\mu = \int \lim_{n\to\infty}\inf f_n \, d\mu$$

Since $f_k \geq g_k$, we get

$$\lim_{k\to\infty}\inf\int f_k d\mu \geq \lim_{k\to\infty}\int g_k d\mu = \int \lim_{n\to\infty}\inf f_n \, d\mu$$

And the result is proved.

Lebesgue's Dominated Convergence Theorem : Assume $g: R^d \to \bar{R}$ is a nonnegative integrable function and $\{f_n\}$ is a sequence of measurable function converging pointwise to f. if $|f_n| \leq g$ for all n, then

$$\lim_{n\to\infty}\int f_n d\mu = \int f \, d\mu$$

Proof : if $\{g - f_n\}$ and $\{g + f_n\}$ be two sequence of nonnegative measurable function, from Fatou's Lemma we have

$$\lim_{n\to\infty}\inf\int (g - f_n) d\mu \geq \int \lim_{n\to\infty}\inf (g - f_n) \, d\mu = \int (g - f) d\mu = \int g \, d\mu - \int f \, d\mu$$

And

$$\lim_{n\to\infty}\inf\int (g + f_n) d\mu \geq \int \lim_{n\to\infty}\inf (g + f_n) \, d\mu = \int (g + f) d\mu = \int g \, d\mu + \int f \, d\mu$$

On the other hand,

$$\lim_{n\to\infty}\inf\int (g - f_n) d\mu = \int g \, d\mu - \lim_{n\to\infty}\sup\int f_n d\mu$$

And

$$\lim_{n\to\infty}\inf\int (g + f_n) d\mu = \int g \, d\mu + \lim_{n\to\infty}\inf\int f_n d\mu$$

Combining the above equations for $\lim\ \inf_{n\to\infty} \int (g - f_n) d\mu$

$$\int g\, d\mu - \lim_{n\to\infty} \sup \int f_n d\mu \geq \int f\, d\mu$$

Hence

$$\lim_{n\to\infty} \sup \int f_n d\mu \leq \int f\, d\mu$$

Similarly for $\lim\ \inf_{n\to\infty} \int (g + f_n) d\mu$, we get

$$\lim_{n\to\infty} \inf \int f_n d\mu \ \geq \int f\, d\mu$$

Hence,

$$\lim_{n\to\infty} \sup \int f_n d\mu \leq \int f\, d\mu \leq \lim_{n\to\infty} \inf \int f_n d\mu$$

i.e.

$$\lim_{n\to\infty} \int f_n d\mu = \int f\, d\mu$$

And the theorem is proved

LEBESGUE AND BOCHNER MEASURABLE FUNCTIONS :

A family of sets V of an abstract space X is called a pre-ring if the following conditions are satisfied: (a) if A_1, $A_2 \in V$, then $A_1 \cap A_2 \in V$, (b) if A_1, $A_2 \in V$, then there exist disjoint sets $B_1, \ldots, B_k \in V$ such that $A_1 \setminus A_2 = B_1 \cup \ldots \cup B_k$. A non-negative finite function v on the pre-ring V is called a positive volume if it satisfies the following condition: for every countable family of disjoint sets $A_t \in V$ ($t \in T$) such that $A = \bigcup_T A_t \in V$ there is $v(A) = \sum_T v(A_t)$.

The triple (X, V, v) will be called a volume space.

Let N be a family of all sets $A \subset X$ such that for every $\varepsilon > 0$ there exists a countable family $A_t \in V$ ($t \in T$) such that $A \subset \bigcup_T A_t$ *and* $\sum_T v(A_t) < \varepsilon$. Sets of the family N will be called null-sets. This family represents a sigma-ideal of sets, that is, it has the following properties: if $A \in N$ then $\cap A \in N$, and given any countable family of sets $A_t \in N$ ($t \in T$) then also $\bigcup_T A_t \in N$.

A condition C (x) depending on a parameter $x \in X$ is said to be satisfied almost everywhere (a. e.) if there exists a set $A \in N$ such that the condition is satisfied at every point $x \notin A$. Let

R be the space of all reals and Y be a Banach space. The norms of the elements of the spaces R, Y will be denoted by | |. Denote by S (Y) the set of all functions being finite sums of functions of the form XAY(Aϵ V, y ϵ Y), where XA is the characteristic function of the set A. This family of functions will be called the set of simple functions. Defined are the formula $\|f\| = \int |f|dv$, where |f| denotes the function |f|(x)=|f(x)| for x ϵ X.

Let a be a function defined by the formula

$$a(\mathcal{Y}) = (1 + |\mathcal{Y})|^{-1}\mathcal{Y} \text{ for } \mathcal{Y}\epsilon Y.$$

Denote by V_0 the family of all sets of the form $A = \bigcup_T A_t$, where A_t ϵ V (t ϵ T) is a countable family of sets.

Let M (Y) denote the space of all functions f from the set X into the space Y such that: (a) f(x) = 0 if $x \notin A$ for some A ϵ V_0 (such a set A will be called a support for the function f), (b) the function $a \circ f$, defined by

$(a \circ f)(x) = a(f(x))$ for $\in X$, is summable on every set A ϵ V. A function g is summable on a set A if XAg ϵ L (Y) . The integral on the set is defined by $\int_A gdv = \int \chi_A gdv$ For measurable functions define the following family of functionals $\|f\|_A = \int_A |a \circ f|dv$ for $A \in V$.

Theorem 7:

(1) The set S (Y) of simple functions is contained in M (Y).

(2) I f_n ϵ M (Y) and f_n (x) → f(x) a.e. then f ϵ M (Y) and $\|f\|_A \to 0\, for$ every A ϵ V.

(3) The following conditions are equivalent:

(a) the function f belongs to the space M (Y),

(b) there exists a sequence of disjoint sets A_m ϵ V such that $f(x) = 0$ if $x \notin \cup A_m$ and the function a o f is summable on every set A_m (m = 1, 2 , . . .) ,

(c) there exists a sequence of simple functions s_n ϵ S (Y) such that $s_n(x)$ → f(x) a.e.

(4) The space M (Y) is linear.

(5) Let W, Y_i (i = 1, 2. . . . . k) be B-spaces and u be a continuous operator $from\ Y_1 \times \dots \dots \times Y_k$ into W such that u(O,. . . . .,O) = 0 . if f_i ϵ M(Y_i) then g - u(f_1,…..,f_k) ϵ M(W), where g(x) = u(f_1(x). . . . . f_k(x)) for x ϵ X.

(6) If Y is a B-algebra then M (Y) is an algebra. Moreover, if f ϵ M (Y), Y has the unity, and g (x) = $(f(x))^{-1}$ a.e. then g ϵ M (Y).

Proof. Since a(0) = 0 the composition a o s represents a simple function. By the definition every simple function is summable on any set A ϵ V. Therefore every simple function is measurable.

Lets prove Part (2). Let A_n ϵ V_0 be supports of the fnctions f_n ϵ M (Y).

Lot B ϵ N be a null-set such that $f_n(x) \to f(x)$ if $x \notin B$ From the definition of a null-set there exists a set A_0 ϵ V_0 such that $\subset A_0$. Put $A = \bigcup_{n=0}^{\infty} A_n$. It can be seen that A ϵ V_0 and the set is a support for the function f. Take any set A ϵ V. The sequence of functions $\chi_A a \circ f_n$ converges almost everywhere to the function $\chi_A a \circ f$ and is dominated by the summable function X_A. Hence $\chi_A a \circ f \in L(Y)$. The sequence of functions $\chi_A |a \circ (f_n - f)|$ converges almost everywhere to zero and is dominated by the function X_A. Therefore,

$\|f_n - f\|_A = \int_A (f_n - f) \, dv \to 0.$

Now lets prove Part (3). Assume that the condition (a) is satisfied and let A = U A_m be a support of the function, where A_m is a sequence of sets from the pre-ring V. It follows from the properties of a pre-ring that one may assume that the sets A_m are disjoint. From the definition of the space M (Y) the function a o f is summable on each A_m. Thus the condition (b) is proven.

Following lemma is needed.

Lemma. If A ϵ V, $X_A f$ ϵ L(Y) , and |f(x)| < 1 a.e., then there exists a sequence of simple functions s_n ϵ S (Y) such that $|s_n(x)| < 1$ if $x \in A, s_n(x) = 0$ if x ϵ A (n = 1, 2 , . . .) and $s_n(x) \to \chi_A(x) f(x)$ a.e.

Proof. Let r_n be a sequence of non-negative numbers increasingly convergent to 1. Let a_n be mappings of the space Y into itself defined by the formula

$a_n(\mathcal{Y}) = \mathcal{Y}$ if $|\mathcal{Y}| \leqq r_n$ and $a_n(\mathcal{Y}) = r_n |\mathcal{Y}|^{-1} \mathcal{Y}$ if $|\mathcal{Y}| > r_n$

From the definition of the space of summable functions there exists a sequence of simple functions $\bar{s}_n \in S\,(Y)$ such that $\bar{s}_n(x) \to X_A(x)$ a.e. It is easy to see that the sequence s_n = $X_A a_n$ o s_n has the required properties. Assume that the condition (b) is satisfied. From the above Lemma to each set A_n and to the function a o f corresponds a sequence s_{mn} of simple functions satisfying the condition of above . Since the inverse of the function a is given by the formula

$a^{-1}(\mathcal{Y}) = (1 - |\mathcal{Y}|)^{-1} \mathcal{Y}$ for $\mathcal{Y} \in Y,\ |\mathcal{Y}| < 1,$

therefore the functions $\bar{s}_{mn} = a^{-1} \circ s_{mn}$ are well defined and belong to the set S(Y). It can be seen that

$\bar{s}_{mn}(x) \underset{n}{\rightarrow} \chi A_m(x) f(x)$ a.e $(m = 1,2, \dots)$

Putting $s_n = \bar{s}_{1n} + \cdots + \bar{s}_{nn}$,

There is $s_n(x) \rightarrow f(x)$. Now assume that the condition (e) is satisfied. From Part (1) and (2) we get that the condition (a) also is satisfied, that is f ϵM (Y) .

Part (4) of the theorem follows from the linearity of the set S (Y) and from Part (3).

To prove Part (5) of the theorem let us take a sequence of simple functions s_{in} ϵ S(Y_i) convergent almost everywhere to the function f_i (i = 1. . . . . k). It is easy to see that the functions u(s_{1n},. . . . ., s_{kn}) are simple and they converge almost everywhere to the function u(f_1, ...,f_k) = g. Thus g ϵ M(W).

The first statement of Part (6) follows from the fact that the operation of multiplication $u(\mathcal{Y}_1, \mathcal{Y}_2) = \mathcal{Y}_1, \mathcal{Y}_2$ for $\mathcal{Y}_1, \mathcal{Y}_2 \in Y$ is a bilinear continuous operator and thus satisfying assumptions of Part (5). To prove the second statement let G denote the set of all y ϵ Y for which there exists the inverse y^{-1}. Define a function b by the formula $b(\mathcal{Y}) = \mathcal{Y}^{-1}$ if $\mathcal{Y} \in G$ and $b(\mathcal{Y}) = 0$ if $\mathcal{Y} \notin G$. Let $s_n \in S(Y)$ be a sequence convergent almost everywhere to the function f. It can be seen that $s_n = b \circ \bar{s}_n \in S(Y)$. Since the set G is open and the function b is continuous on it, we have $s_n(x) \rightarrow g(x) = (f(x))^{-1}$ a.e. Thus the theorem is proven.

Denote by $L^+(S^+)$ the set of all functions such that f ϵ L(R)[S(R) respectively] and f(x) ≥0 for all x ϵ X. Let c be the function defined by the formula

$c(\mathcal{Y}) = (1 + \mathcal{Y})^{-1}\mathcal{Y}$ for $\mathcal{Y} \in \langle 0, \infty)$ and $c(\infty) = 1$.

This function establishes a homeomorphism of the interval $\langle 0, \infty\rangle$ onto the interval $\langle 0,1\rangle$. Let M^+ be the set of all functions f from the space X into the interval$\langle 0, \infty\rangle$ such that the function f has a support $A \in V_0$and the composed function $c \circ f$ is summable on every set $A \in V$. The functions of M^+ will be called nonnegative extended measurable functions.

Theorem 8:

(1) The set S^+ of non-negative simple functions is contained in M^+.

(2) Let f be a function from X into $\langle 0, \infty\rangle$ and let $f_n \in M^+$./, If $f_n(x) \rightarrow f(x)$ a.e. then $f \in M^+$.

(3) Let f be a function from X into $\langle 0, \infty\rangle$ then the following conditions are equivalent:

(a) the function f belongs to the space M^+,

(b) there exists a sequence of disjoint sets $A_m \in V$ such that $f(x) = 0$ if $x \in \cup A_m$ and the function $c \circ f$ is summable on every set A_m,

(c) there exists a sequence of simple functions $s_n \in S^+$such that $s_n(x) \rightarrow f(x) a.e.$

(4) The set L^+ of non-negative summable functions is contained in M^+.

(5) A function f from X into $\langle 0, \infty \rangle$ belongs to the space M^+ iff there exists an increasing sequence of functions $f_n \in L^+$ such that $f_n(x) \to f(x)$ a.e.

(6) The set M^+. forms a positive multiplicative cone, that is, if $f_1, f_2 \in M^+$ and $t_1, t_2 \in \langle 0, \infty \rangle$ then $t_1, t_1 + t_2, t_2 \in M^+$ and $f_1, f_2 \in M^+$.

(7) If $f \in M(Y)$ then $|f| \in M^+$.

Proof. The proof of Part (1) is obvious. The proofs of Parts (2) and (3) are similar to the proofs of Part (2) and (3) of above Theorem . Let us prove Part (4). Take any function $f \in L^+$. From the definition of the space $L\ (R)$ we get there exists a sequence of simple functions $\bar{s}_n \in S(R)$ such that $\bar{s}_n(x) \to f(x)$ a.e. We see that $s_n = |\bar{s}_n| \in L^+$s and $s_n(x)|f(x)| = f(x)$ a.e. Now from Part (1) and (2) of the theorem we conclude that $f \in M^+$. To prove Part (5) take any function f ϵ M+. Let s_n ϵ S^+ be a sequence of simple functions such that $s_n(x) \to f(x) if x \in A \in N$ Put $f_n(x)=\inf\{s_m(x) : m \geq n\}$ for xϵ X. Have f_n ϵ L^+ according to previous theorems. It is easy to see that if x ∉ A then the sequence f_n (x) is convergent to the value f(x). Conversely, if f_n ϵ L^+ and $f_n(x) \to f(x)$ a.e. then from Parts (2) and (4) of the theorem we get f ϵ M^+. The proof of Part (6) follows from Part (3) and is similar to the proof of Part (4) and (5) of above Theorem.

To prove Part (7) it is enough to notice that if $s_n(x) \to f(x)$ a.e. then $|s_n|$ ϵ S^+ and $|s_n|(x) \to |f|(x)$ a.e. Thus the theorem is proven.

LEBESGUE MEASURE

Having constructed the outer measure μ^* and explored its basic properties, we are now ready to define the measure μ.

Definition *The* Lebesgue measure μ *is the restriction of the outer measure* μ^* *to the measurable sets, i.e. it is the function*

μ : M→ [0, ∞]

defined by $\mu(A) = \mu^*(A)$

for all $A \in$ M.

Remark: Since μ and μ^* are essentially the same function, you may wonder why we have introduced a new symbol for the Lebesgue measure. The answer is that although it is not going to make much of a difference for us here, it is convenient to distinguish between the two in more theoretical studies of measurability. All you have to remember , is that $\mu(A)$ and $\mu^*(A)$ are defined and equal as long as A is measurable.

We can now prove that μ has the properties we asked for at the beginning of the chapter:

Theorem 9 : The lebesgue measures μ : M→ [0, ∞] has the following properties :

(i) μ(ø) = 0

(ii) Assume that A ϵ M and μ(A) = 0. Then all subset $B \subset A$ are measurable and μ(B) = 0. (Completeness)

(iii) *if* $\{A_n\}_{n \epsilon N}$ is a sequence of measurable sets, then [countable sub-additivity]

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

(iv) [countable additivity] *if* $\{E_n\}_{n \epsilon N}$ is a disjoint sequence of measurable sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

(v) For all closed boxes

$$B = \left[b_1^{(1)}, b_2^{(1)}\right] \times \left[b_1^{(2)}, b_2^{(2)}\right] \times \dots \dots \dots \times \left[b_1^{(d)}, b_2^{(d)}\right]$$

We have

$\mu(B) = |B| = \left[b_2^{(1)} - b_1^{(1)}\right]\left[b_2^{(2)} - b_1^{(2)}\right] \dots \dots \dots \left[b_2^{(d)} - b_1^{(d)}\right]$

Proof : (i) (ii) and (ii) can be proved through above lemma and (v) follows (iv) so in order to prove (iv) we see that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

By (iii). To get the opposite inequality

$$\sum_{n=1}^{N} \mu(E_n) = \mu\left(\bigcup_{n=1}^{N} E_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

Since this holds good for N ϵ **N** we have

$$\sum_{n=1}^{\infty} \mu(E_n) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

Hence both have inequalities and (iii) is proved

Lemma : If C, D are measurable ssets such that $C \subset D$ and $\mu(D) < \infty,$ then

$$\mu(D \backslash C) = \mu(D) - \mu(C)$$

Proof: By additivity

$$\mu(D) = \mu(C) + \mu(D \backslash C)$$

Since $\mu(D)$ is finite, so if $\mu(C)$,and it make sense to subtract $\mu(C)$ on both sides to get

$$\mu(D \backslash C) = \mu(D) - \mu(C)$$

The next properties are often referred to as continuity of measure:

Proposition : Let $\{A_n\}_{n \in N}$ be sequence of measurable sets.

(i) If the sequence is increasing $(i.e. A_n \subset A_{n+1}\ for\ all\ n)$,then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

(ii) If the sequence is decreasing like $(i.e. A_n \supset A_{n+1}\ for\ all\ n)$ and $\mu(A_n)$ is finite, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

Proof : (i) If we put $E_1 = A_1$ and $E_n = A_n \backslash A_{n-1}$ for $n > 1$, the sequence $\{E_n\}$ is disjoint, and $\bigcup_{k=1}^{n} E_k = A_n$ for all N. Hence

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} E_k\right) = \lim_{n\to\infty} \mu(A_n)$$

Where we have used the additivity of μ twice.

(iii) We first observe that $\{A_1 \backslash A_n\}_{n\in N}$ is an increasing sequence of sets with union $A_1 \backslash \bigcap_{n=1}^{\infty} A_n$. By part (ii), we thus have

$$\mu\left(A_1 \backslash \bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu\left(\frac{A_1}{A_n}\right)$$

Applying previous lemma on both side

$$\mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty}\big(\mu(A_1) - \mu(A_n)\big) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$

Cancelling, we get

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

As we set out to prove.

Example (a) : we already know that the closed boxes are "right" measures, what about open boxes ? if

$$B = \left[b_1^{(1)}, b_2^{(1)}\right] \times \left[b_1^{(2)}, b_2^{(2)}\right] \times \dots\dots\dots \times \left[b_1^{(d)}, b_2^{(d)}\right]$$

is an open box and let B_n be the closed box

$$B_n = \left[b_1^{(1)} + \tfrac{1}{n}, b_2^{(1)} - \tfrac{1}{n}\right] \times \left[b_1^{(2)} + \tfrac{1}{n}, b_2^{(2)} - \tfrac{1}{n}\right] \times \dots\dots . \times \left[b_1^{(d)} + \tfrac{1}{n}, b_2^{(d)} - \tfrac{1}{n}\right]$$

Obtained by moving all wall a distance 1/n inward, by above preposition we have

$$\mu(B) = \lim_{n\to\infty} \mu(B_n)$$

And since B_n have "right" measure, the result follows.

Example (b) : Let

$$K_n = [-n, n]^d$$

Be the closed box centered at the origin and with edges of length $2n$. For any measurable set A , it follow from the proposition above that

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap K_n)$$

We shall need one more property of measurable sets. It tells us that measurable sets can be approximated from the out side by open sets and from the inside by closed sets.

Proposition Assume that $A \subset R^d$ is a measurable set. For each $\epsilon > 0$, there is an open set $G \supset A$ such that $\mu(G\backslash A) < \epsilon$ and a closed set $F \subset A$ such that $\mu(A\backslash F) < \epsilon$.

Proof : we begin with the open sets. Assume first A has finite measure. Then there is a covering $\{B_n\}$ of A by open rectangles such that

$$\sum_{n=1}^{\infty} |B_n| < \mu(A) + \varepsilon$$

Since $\mu(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} |B_n|$,we see that $G = \bigcup_{n=1}^{\infty} B_n$ is an open such that $A \subset G$, and $\mu(G) < \mu(A) + \epsilon$. Hence

$$\mu(G\backslash A) = \mu(G) - \mu(A) < \varepsilon$$

If $\mu(A)$ is infinite , we first use the boxes K_n in Example (b) to slice A into pieces of finite measure. More precisely , we let $A_n = A \cap (K_n\backslash K_{n-1})$, and use what we have already provide to find an open set G_n such that $A_n \subset G_n$ *and* $\mu(G_n\backslash A_n) < \frac{\varepsilon}{2^n}$. Then is an open set which contains A, and since $G\backslash A \subset \bigcup_{n=1}^{\infty}(G_n\backslash A_n)$, we get

$$\mu(G\backslash A) \leq \sum_{n=1}^{\infty} \mu(G_n\backslash A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

And proving about the approximation by open sets. To prove the statements about closed sets let us apply the first part of theorem to A^c to get an open set $G \supset A^c$ such that $\mu(G\backslash A^c) < \varepsilon$ That means that $F = G^c$ is a closed set such that $F \subset A$, and since $A\backslash F = G\backslash A^c$, we have

$$\mu(A\backslash F) < \varepsilon$$

We have now thus established the basic properties of Lebesgue

CONCLUSION :

This paper understands, in-depth and reviews the Lebesgue's Measurable Function, Lebesgue's Measures and its integration. Though not elaborated, this paper limits the scope of its study to non-negative functions.

This paper also efforts a study of a unified theory for *d*-dimensional volume based on the notion of a *measure*, and have tried to use this theory to build a stronger and more flexible theory for integration.

It was observed that the questions studied in this paper have simple answers at the recursive level. It is hoped that the study of this paper not only increases our understanding of the intrinsic difficulty of various numerical operations, but also provides new insight.

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